DISTRIBUTED CONTENT COLLECTION AND RANK AGGREGATION

BY

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DISSERTATION

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ABSTRACT

Despite the substantial literature on recommendation systems, there have been few studies in distributed settings, where peers provide recommendations locally. Motivated by word of mouth type of social behavior and the advantages of sharing resources, we analyze an online distributed recommendation system with joint content collection and rank aggregation. In such a system, peers contact each other and exchange partial preference information about items, which, for example, could be videos. Peers use recommendation strategies to make decisions with limited knowledge and collect items that are available from the contacted peers. The goal is to maximize the rate at which peers collect their most preferred items.

Correlated preferences are modeled as rankings generated by a Plackett-Luce ranking model with Zipf popularity distribution. We establish a performance upper bound and use intuition provided by the bound to design recommendation strategies with a range of complexity. Among these, the direct recommendation rule emerges as being particularly simple and yet effective. The direct recommendation rule is found to be remarkably robust, working well over a broad range of correlation of preferences, initial video availability, storage size, peer arrival pattern, and performance metric.

Correlated preferences are modeled as scores generated using an independent crossover model. In order to explore performance for large scale networks, we identify the fluid limit as the number of videos goes to infinity for a mean field limit derived for the number of peers going to infinity under a direct recommendation rule. Simulation results show that the limit analysis accurately predicts performance, not only for the independent crossover model with scores, but also a model with rankings. The performance of the direct recommendation rule is shown to be near optimal for large scale systems.

Correlated preferences are modeled as scores generated using a two-stage independent crossover model. We propose four recommendation strategies for heterogeneous preferences. We find that a simple rule, called the nearest stored preference rule, is as effective as the more complex rules. The performance of all the rules is far from a performance upper bound in case the peers in different clusters are nearly independent. We find through simulation that the gap can be nearly closed by using either exponential accumulation of information or neighbor assignments such that most neighbors have similar preferences.

To my parents Yang Yang and Maizhi Yang
and to my wife Congcong Li,
for their love and support.

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CHAPTER 1

INTRODUCTION

Online social networks have become a big part of our lives. They are especially helpful in advising people how to make decisions. In particular, there is a rapidly growing number of objects and growing amount of information describing the objects. Each person has somewhat different interests and needs to search for the most relevant objects. Fortunately, in a given class of objects, people's preferences are statistically positively correlated, even if they have some disagreements. This allows for a crowd sourcing effect, where people can help each other find the most suitable objects. In this thesis, we take the objects to be videos.

In a traditional video recommendation system, recommendations are provided in a centralized way. Peers upload preference information to a central server and recommendations are computed by the central server. Furthermore, peers only download videos from the server. In contrast, we examine an online system in which peers are only contacting each other, with, perhaps, some centralized coordination such as a tracking system. Peers collect preference information only from their contacted peers and provide recommendations for each other based on limited knowledge. Then, peers download videos from their contacted peers. The availability of videos depends on peers' past contacts and decisions. Well designed peer-to-peer systems can require less computational complexity, save server bandwidth and storage, and resist malicious recommendations made by a server or small group of peers.

The objective is to have each peer efficiently collect as many personally preferred videos as possible under constraints on bandwidth, storage, and information about preferences. A main factor contributing to efficiency is the choice of recommendation rule that helps peers decide which videos to collect. Strategies differ in how much preference information is gathered, stored, and processed.

1.1 Motivation

Word of mouth plays an important role in how people make everyday decisions—for example, choosing from candidates applying for a job position, restaurants in a city, books to read, music to listen to, etc. The setting of our work is movie videos. Having the peers receive recommendations in a distributed way could be more helpful than receiving recommendations from a centralized authority. Good decisions naturally lead to satisfaction, happiness, and sometimes progress. However, not all choices are equally good or bad because people have different views. Yet, it is crucial for people to be efficient in making good decisions because time is a valuable resource. Since people inherently have similar tastes, we need to find recommendation strategies to take advantage of this correlation so peers can be efficient in making good decisions.

1.2 Related Work

A wide variety of recommendation systems have been proposed, mostly based on a centralized system in which recommendations are generated by a centralized authority. Some recommendation systems may use matrix completion techniques offline [1]. Online recommendation systems often use neighborhood-based collaborative filtering that falls under peer-peer or item-item paradigms, in which either an item is recommended to similar peers or similar items are recommended to a peer [2, 3, 4, 5, 6]. Another interesting aspect of online recommendation systems is cold-start recommendations, where many peers or many items are added to the system at once. In this regime, network structures such as social networks are often used [7].

This thesis pertains to an online distributed recommendation system with content collection and rank aggregation done jointly, starting primarily from a cold start state. Basic elements of the model were proposed in the pioneering work of Cruz [8]. Cruz modeled correlated preferences by assuming there is a master ranking of videos, and the personal preferences of a peer are correlated with the master ranking, giving rise to correlated personal preferences among peers. Cruz studied the content collection problem by formulating a happiness function to measure how many personally preferred videos each peer has collected. He proposed a solution to the joint rank aggregation problem based on each peer compiling a lossy aggregate of partial rankings called the *global list* maintained by a peer, to be used to assist in selecting which videos to download.

The choice of rank aggregation method is a major aspect of the system. There is a large literature on rank aggregation; here we cover the part most relevant to this thesis. In our work, we assume the observed preferences are generated from a probabilistic model. There are many probabilistic models on permutations, some of which are studied in [9] and [10], which fall into one of two categories: nonparametric models and parametric models. Jagabathula and Shah [11] studied a nonparametric approach to modeling distributions over rankings. Most of the parametric models fall into one of the following three categories: noisy comparison model, distance-based model, and random utility model. Our work is based on the random utility model.

The parametric models assume there is an underlying master ranking σ over m objects. For the noisy comparison model, each peer independently gives a pairwise comparison which agrees with the master ranking with probability $p > \frac{1}{2}$ [12]. An example of distance-based model is the Mallows model, which randomly generates a full ranking σ over m objects from the master ranking σ^* with probability proportional to $e^{\beta d(\sigma,\sigma^*)}$, where β is a fixed spread parameter and $d(\cdot,\cdot)$ can be any permutation distance such as the Kemeny distance [12].

In our work, we adopt a special case of random utility model (RUM) known as the Plackett-Luce (PL) model. Plackett [13] performed analysis on permutations and Luce [14] formulated theory on individual choice behavior. The PL model is assumed in many ranking related works. The Bradley-Terry model [15] is a special case of the Plackett-Luce model obtained when only pairwise partial rankings are reported. Hunter [16] showed that the log likelihood function under the PL model is concave and proposed a minorize/maximize (MM) algorithm for rank aggregation under generalized Bradley-Terry models. He also showed that the algorithm may be problematic for sparse data due to over fitting. Guiver and Snelson [17] proposed a Bayesian inference method for the PL model. This method uses divergence measure and message passing techniques proposed by Minka [18]. In the work of Cao et al. [19], rank aggregation is done by minimizing a particular metric.

A well known probabilistic model of Thurstone [20] states a law of comparative judgment which specifies an unobserved random variable x_i for each object according to a distribution. It is well known that Luce's choice axiom is equivalent to the Thurstone model using Gumbel distributed independent noise, a result attributed to Holman and Marley (see Lemma 6 of [21]).

There are some relations between our work and the theory of opinion dynamics. The focus of opinion dynamics is on distributed averaging for consensus problems. Boyd et al. [22], Nedich [23] and many others have studied the averaging problem under various gossip constraints for different network graphs. The estimate of the master ranking, in particular the global list, is similar to opinion dynamics because the global list acts as a consensus building mechanism. The global list's prediction result depends on the weighting function just as the consensus depends on the model of the opinion dynamics. The model we study is different from those in much of the literature on opinion dynamics because peers do not know their opinions before watching the videos, so the opinions of peers are driven initially by the opinions of others and ultimately by their personal evaluations.

1.3 The Basic Peer-to-Peer Framework

We explain in this section the assumptions about how peers contact each other to exchange videos. We consider a simple closed homogeneous system. The number of peers, N, and the number of videos, M, are fixed over time. The peers are assumed to be indexed by $[N] \triangleq \{1, \ldots, N\}$ and the videos are assumed to be indexed by $[M]$. It is assumed that there is a master preference for the videos, and the preferences of individual peers are noisy versions of the master preference. In the case preferences are expressed as rankings, there is a master ranking of videos, and the rankings of peers are generated independently using the Plackett-Luce distribution with parameters ordered according to the master ranking, giving rise to the Plackett-Luce (PL) model. In case preferences are expressed as scores, a master score is assigned to each video, and the personal scores of peers are generated independently using a crossover probability matrix, giving rise to the independent crossover (IC) model. There are L possible scores, $[L] = \{1, ..., L\}$, for each video. Let

$G_1, \ldots, G_L$ be disjoint subsets of $[M]$, with union equal to $[M]$, and call the videos indexed by G_ℓ *type* ℓ *videos*. This partition is equivalent to giving a master score vector, where the video types are master scores. Let $m_\ell = |G_\ell|$ for $\ell \in [L]$. We adopt the convention that a lower numerical rank or score indicates a more preferred video.

We imagine that the peers have no a priori information about which videos they will eventually prefer, so the ordering of the videos according to master preference is assumed to be uniform over all $M!$ permutations of the videos. But for ease of notation in performance analysis, we *assume without loss of generality that the videos are indexed in order of nonincreasing master preference.*[1]

The system proceeds in synchronized time steps. Initially each peer is endowed with C videos, assumed to be chosen uniformly at random from among the set of M videos. Each video in the system occupies one unit of storage space. Each peer in each time slot connects to another peer (called the contacted peer) selected uniformly at random in order to download (i.e. pull) a video, after a possible exchange of preference information. The link bandwidth is assumed to be just sufficient to allow each peer to download one video in each time step. The bandwidth incurred from the exchange of preference information is assumed to be negligible.

We assume that each peer can store up to $S_{\max}$ videos for some value of $S_{\max}$. It might be relevant in some situations to consider systems for which storage space imposes a severe constraint on the peers, but the focus of this thesis is on a regime such that the storage constraint is not very tight. In particular, for the parameters we have examined, peers can store all the videos they personally highly prefer. Since peers evaluate the videos they download, we assume that if a peer needs to evict a video, it evicts a personally least preferred one.

A happiness function, $H_n(t)$, is a performance metric used to measure how happy each peer n is at any given time, depending on its personal preference and the set of videos it has collected. We require $H_n(t)$ to take values in $[0, 1]$ and to be a nonnegative, nonincreasing function of the set of ranks or set of scores of the videos; i.e., a set of more favored videos gives higher

[1] A slight drawback of this convention is that in simulations, care must be taken to not make any selection decisions, such as tie-breaking rules, depending on the indices of the videos.

happiness than a set of less favored videos of the same size. The normalized total system happiness at time t is defined to be the average, over all N peers, of the happiness of individual peers. The objective is to have the peers obtain collections of videos favored by themselves as quickly as possible, using estimation and recommendation techniques. That is, to make the normalized total system happiness quickly converge to one.

1.4 Main Contributions

The contributions of this thesis are the following:

Chapter 2: Ranking System

- In the single-cluster regime with homogeneous population, we provide a performance upper bound based on a stochastic ordering property, for distributed content collection and rank aggregation in the context of the PL ranking model originally considered by Cruz [8].

- In the direction of an elaborate recommendation rule, we explain the use and performance in simulation of a rank aggregation based on iterative computation of a maximum likelihood estimator (MLE) by the MM algorithm (similar to EM algorithm) of Hunter [16].

- In the direction of a moderate complexity recommendation rule, we provide variations of Cruz's global list recommendation rule that markedly improves the performance. We find in simulations that the improved versions of Cruz's global list recommendation rule perform close to the upper bound.

- In the direction of a very simple recommendation rule, we propose the *direct recommendation rule*, under which a peer downloads the video most highly recommended by the contacted peer. Simulation results show near optimal performance for this rule, not far behind the performance of the other rules mentioned. Moreover, this rule appears to be robust even with different levels of peer similarities, initial video availabilities, storage sizes, peer arrival rates, and happiness functions.

Chapter 3: Scoring System with Large System Scaling

- We demonstrate how to apply mean field analysis (letting the number of peers go to infinity) followed by fluid analysis (letting the number of videos go to infinity) to provide a simple explicit approximation formula for performance of the IC model under the direct recommendation rule. We show through simulations that the analysis provides an accurate prediction of the uptake rate of different videos by the population of peers, even for the PL model. In addition, the numerical result of the direct recommendation rule appears to be near optimal. The technique for rigorously establishing the fluid limit of the mean field model should be of independent interest to researchers in the network performance analysis area.

- We prove a rigorous connection between the PL ranking model and IC scoring model, in the limit of a large number of videos. This result should be of independent interest to researchers in the ranking area.

Chapter 4: The Multi-Cluster Framework

- In the multi-cluster regime with heterogeneous population, we propose and demonstrate the performance of several recommendation rules based on the insights gained from the study of recommendation rules under the single-cluster model with homogeneous population. We distinguish two aspects of a recommendation rule: accumulation of information and processing of information.

- In the direction of a recommendation rule using substantial information processing, we propose the *Bayesian rule with soft or hard clustering.* The Bayesian rule with soft clustering assigns weight to each stored preference vector and makes use of all preference information for score prediction. In contrast, the Bayesian rule with hard clustering uses a threshold based on similarity and makes use of only highly correlated preference information for score prediction.

- In the direction of a recommendation rule using moderate information processing, we propose the *nearest stored preference recommendation rule*, which involves a peer following a single preference vector from other peers with the highest correlation to make a video selection.

Simulation results show performance similar to that of the Bayesian rules.

- In the direction of a recommendation rule requiring minimal information accumulation, we propose the *multi-cluster aware global list recommendation rule*, which recursively combines preference information. Because of noise introduced in the combining process, we find through simulations that its performance is not as good as the other recommendation rules when peers in different clusters are correlated.

- We show that either exponential accumulation of partial preference vectors or neighbor assignments such that most neighbors have similar preferences is sufficient for any of the above rules using stored preference vectors to yield near optimal performance.

1.5 Organization

Chapter 2 analyzes the ranking system constructed using the PL model in the single-cluster regime with homogeneous population. Chapter 3 analyzes the scoring system constructed using the IC model in the single-cluster regime with homogeneous population. Chapter 4 analyzes the scoring system constructed using the IC model in the multi-cluster regime with heterogeneous population.

CHAPTER 2

RANKING SYSTEM

2.1 Plackett-Luce with Zipf(α) Model

For the PL model, the peers express preferences for videos by rankings, which we model as follows. The videos are assumed to be indexed by $[M]$ in decreasing order of master preference. Each peer n for $n \in [N]$ has an intrinsic personal ranking of videos, $R_n : [M] \to [M]$, which is a permutation of $[M]$. Let $P_n : [M] \to [M]$ be the inverse ranking function of R_n, so that $P_n(r)$ is the index of the video with rank r in R_n. The ranking R_n is a noisy version of the master ranking, and is assumed to have the PL distribution with some parameters $w = (w_1, w_2, \ldots, w_M) \in \mathbb{R}_+^M$ such that $w_1 \geq \cdots \geq w_M$. The probability of a particular personal preference, P_n, is thus given as follows:

$$\mathbb{P}(P_n|w) = \prod_{m\in[M]} \frac{w_{P_n(m)}}{w_{P_n(m)} + w_{P_n(m+1)} + \ldots + w_{P_n(M)}}. \tag{2.1}$$

That is, the distribution corresponds to weighted sampling without replacement, where the weights are the parameters. The most preferred video, $P_n(1)$, is randomly selected with probabilities proportional to the weights. Given the $m-1$ most preferred videos, the m^{th} most preferred video is selected from the remaining $M-(m-1)$ videos with probabilities proportional to weights.

The following *exponential representation* of the PL distribution is well known; it is connected to the fact the PL model is a special case of the Thurstone model [21]. If $X_1, \ldots, X_M$ are independent, exponentially distributed random variables such that X_m has rate parameter w_m for each m, then the rank of the m^{th} video, $R_n(m)$, is equal to the rank of X_m among the M values $X_1, \ldots, X_M$ (with smaller numbers having lower numerical ranks). We assume the w_m's are decreasing in m so that peers tend to prefer lower indexed videos. Note that the PL distribution is invariant with respect to multiplying all the weights by a constant. Since real life demand curves tend to follow a heavy-tailed distribution [24], following Cruz [8], we often assume a Zipf(α) distribution as the parameters for the PL model [24]; $w_m \propto m^{-\alpha}$ for some $\alpha > 0$.

We adopt the following additional notation:

- $S_n(t)$: subset of videos peer n is storing at t^{th} time step
- $V_n(t)$: subset of videos peer n has downloaded (and viewed) by t^{th} time step
- $RS_n^t(m)$ relative rank of video m in $S_n(t)$ determined by peer n's ranking function, R_n
- $RV_n^t(m)$ relative rank of video m in $V_n(t)$ determined by peer n's ranking function, R_n

If A, B are finite subsets of $\mathbb{R}$, $A \succ B$ (A is better than B) indicates that $|A| \geq |B|$ and $a_{[i]} \leq b_{[i]}$ for $1 \leq i \leq |B|$, where $a_{[1]} < \ldots < a_{[|A|]}$ denotes the ordered elements of A and $b_{[1]} < \ldots < b_{[|B|]}$ denotes the ordered elements of B. The happiness function of a peer at time t is defined as $H_n(t) = f(\{R_n(m) : m \in S_n(t)\})$, where $f : 2^{[M]} \to \mathbb{R}$, is assumed to be nondecreasing in the $\succ$ order, and $2^{[M]}$ denotes the set of subsets of $[M]$.

2.2 An Upper Bound on System Performance

To obtain an upper bound on the happiness of a peer, no matter what recommendation rule is used, consider an idealized system in which the peer has access to a server that can provide any video, and for which a genie reveals extra information. If the genie revealed to the peer the peer's own personal rankings of the videos, $(R(m) : m \in [M])$, then the obviously optimal rule of the peer would be to download videos in the order of increasing numerical personal rank. This provides a rather trivial upper bound on the happiness of a peer vs. time. Our focus is on a tighter upper bound, derived by considering the case in which the peer has access to a server, and the genie reveals only the master ranking to the peer. Since the preferences of other peers in the system are conditionally independent given the master ranking, the rankings of other peers provide no additional clues to the peer about its own preferences. The following theorem shows the peer's optimal recommendation rule is to download the videos in the order of increasing master rank (i.e. increasing index m).

Theorem 2.2.1. *Let $f : 2^{[M]} \to [0,1]$ be nondecreasing in the happiness order $\prec$. Let $(R(m) : m \in [M])$ denote a random ranking vector generated by the* PL *model with ordered weight parameters $w_1 \geq \cdots \geq w_M$. If $A, B \subset [M]$ such that $A \succ B$, then $\mathbb{E}\left[f(\{R(m) : m \in A\})\right] \geq \mathbb{E}\left[f(\{R(m) : m \in B\})\right]$.*

Proof of Theorem 2.2.1. We begin by introducing additional notation. If F, G are random finite subsets of $\mathbb{R}$, $F \succ_s G$ ("s" for "stochastic") indicates there exists a pair of random sets $\widetilde{F}, \widetilde{G}$ on one probability space such that (i) $F \overset{d.}{=} \widetilde{F}$, (ii) $G \overset{d.}{=} \widetilde{G}$, (iii) $\mathbb{P}\left\{\widetilde{F} \succ \widetilde{G}\right\} = 1$. Given disjoint sets $F, \widetilde{F} \subset \mathbb{R}$, let $\Gamma(F, \widetilde{F})$ denote the set of ranks of the elements of F among the elements of $F \cup \widetilde{F}$. For example $\Gamma(\{0.2, 0.6, 0.3\}, \{0.4, 1.7\}) = \{1, 2, 4\}$. Using the set order $\succ$, $\Gamma(F, \widetilde{F})$ is nondecreasing in F and nonincreasing in $\widetilde{F}$.

Suppose the assumptions of Theorem 2.2.1 hold. It suffices to prove $\{R(m) : m \in A\} \prec_s \{R(m) : m \in B\}$. We use the exponential representation of the PL distribution, so there exist M independent random variables $(X_m : m \in [M])$ such that X_m is exponentially distributed with parameter w_m and $R(m) = \Gamma(\{X_m\}, \{X_{m'} : m' \in [M] \backslash \{m\}\})$. A key property is that the random variables X_m are stochastically nondecreasing in m.

Observe that $\{R(m) : m \in A\} = \Gamma(\{X_m : m \in A\}, \{X_m : m \in [M] \backslash A\})$ and the two arguments of Γ in this instance are mutually independent. Similarly, $\{R(m) : m \in B\} = \Gamma(\{X_m : m \in B\}, \{X_m : m \in [M] \backslash B\})$, where, again, the two arguments of Γ are independent. The fact that $A \succ B$ and the X's are independent and stochastically ordered implies $\{X_m : m \in A\} \succ_s \{X_m : m \in B\}$ and $\{X_m : m \in [M] \backslash A\} \prec_s \{X_m : m \in [M] \backslash B\}$. The conclusion follows from the monotonicity property stated for Γ. $\square$

In words, if a set A of videos is better than a set B of videos in the master order, then A is stochastically better than B for a peer. The bound suggests that performance in the original system would be nearly optimal if (1) the peers could quickly and accurately infer the master order by sharing preference information, and (2) the peer-to-peer content distribution mechanism could provide requested videos nearly as well as a centralized server.

2.3 Three Recommendation Rules

Three rules are presented in this section that could be used by peers to determine which videos to download. The first two seek to aggregate rankings, and then the peer downloads the video it does not have that is estimated to be most liked. The third rule, the direct recommendation rule, uses a minimal amount of state information.

2.3.1 EM Algorithm Rank Aggregation and MLE Recommendation Rule

Section 2.2 suggests that peers should try to download videos in the master ranking order. We describe a fairly elaborate estimation technique in this section that the peers could use to estimate the master preference order. The idea is for each peer to accumulate all the partial personal rankings of the peers it contacts, and apply a rank aggregation algorithm to estimate the master ranking.

A peer n does not initially know its own personal ranking, but it is assumed that after viewing videos, it can determine the ordering of those videos among themselves, consistently with their order in the personal ranking of the peer. Thus, eventually, if a peer views all of the videos, it can discover its own personal ranking vector. We use RV_u^t to denote the partial ranking a peer u has determined for the videos V_u^t that it has viewed up until time t.

Let $K_u(t) \in [N]$ be the peer that peer u contacts at the t^{th} iteration. In each time step, each peer u pulls the partial ranking from its randomly contacted peer $v = K_u(t)$, i.e. $RV_v^t(m)$ for all $m \in V_v(t)$. Peer u adds peer v's partial ranking to the list of its previously gathered partial rankings, denoted by $Y = \{RV^1_{K_u(1)}, RV^2_{K_u(2)}, ..., RV^t_{K_u(t)}\}$. As shown by Hunter [16], there are efficient algorithms to compute the maximum likelihood estimator (MLE) of the parameters w_m for the videos from a list of partial permutations generated by the PL model. In particular, the log likelihood ratio is concave in $\log(w_m)$ and the iterative MM algorithm, or the closely related EM algorithm [25], can be used for the computation. The EM algorithm is shown below; it is very similar to the MM algorithm of [16]. Note that while we call the estimator the MLE estimator, it only maximizes likelihood under the false assumption that the sets of videos observed have nothing to do with the rankings of the videos. The assumption is false because the other peers have decided which videos to view based on ranking information.

On one hand, this method does not even use all the information about rankings that peers could collect under the rules of engagement for the joint content collection and rank aggregation model we have assumed. In particular, the contacted peers could pass on not only their own personal rankings of the videos they have collected, but also information the contacted peers have gathered from others. In this way, the information available to a peer could be growing exponentially in time.

On the other hand, as we shall see in simulations, the information that is shared for this rule leads to performance very close to the upper bound of Section 2.2 in simulations, so there is little motivation to consider collecting even more information. Rather, it seems more interesting to see how well the peers can do while maintaining less information, which is the motivation behind the recommendation rules considered next.

EM Algorithm for Estimating the Master Ranking

To derive an EM algorithm we use the exponential representation of the PL distribution, with a vector of X's becoming the complete data for each observed partial ranking, and the order of the X's being the observed data. The complete data available at peer u at time t can thus be expressed as $X = \left(\overline{X}_{K_u(1)}, \overline{X}_{K_u(2)}, ..., \overline{X}_{K_u(t)}\right)$ where $\overline{X}_{K_u(s)} = \left(X_{K_u(s),m} : m \in V_{K_u(s)}(s)\right)$. Each random variable of the form $X_{v,m}$ is exponentially distributed with rate parameter $exp(\theta_m)$.

Recall $Y = \left(RV^1_{K_u(1)}, RV^2_{K_u(2)}, ..., RV^t_{K_u(t)}\right)$ is the vector of observed partial rankings. Let $PV^i_{K_u(s)}(r) : [|V_{K_u(s)}(s)|] \to V_{K_u(s)}(s)$, be the partial preference function, which is the inverse of $RV^i_{K_u(s)}$. Denote the conditional probability of the complete data given $\overline{\theta}$ by $\mathbb{P}_{cd}(X|\overline{\theta})$. The EM algorithm at the t^{th} iteration is shown as follows:

The expectation step is

$$Q(\overline{\theta}|\overline{\theta}^{(k)}) \quad = \quad E\left[\log \mathbb{P}_{cd}(X|\overline{\theta})|Y, \overline{\theta}^{(k)}\right].$$

The maximization step is to select $\overline{\theta}^{(k+1)} = \arg\min_{\overline{\theta}} Q(\overline{\theta}|\overline{\theta^{(k)}})$, which can be computed in closed form, yielding the following iteration formula:

$$\theta_m^{(k+1)} = \frac{|\{i|m \in V_{K_u(s)}(s), 1 \le s \le t\}|}{\sum_{s:m \in V_{K_u(s)}, 1 \le s \le t} E\left[X_{(K_u(s),m)}|RV^s_{K_u(s)}, \overline{\theta}^{(k)}\right]}, \tag{2.2}$$

where
$E\left[X_{(K_u(s),m)}|RV^s_{K_u(s)}, \overline{\theta}^{(k)}\right] = \sum_{j=1}^{RV^s_{K_u(s)}(m)} \left(\frac{1}{\sum_{m \in V_{K_u(s)}(s)} \theta_m^{(k)} - \sum_{r=1}^{j-1} \theta^{(k)}_{PV^s_{K_u(s)}(r)}}\right)$.

Then, the order of video selection follows the decreasing order of θ's. The derivation is shown in the end of this chapter.

2.3.2 Cruz's Global List Recommendation Rule for Rank Aggregation

Instead of storing all the partial rankings it receives from other peers as in the previous section, a peer u can aggregate the information using a state for each video, to reduce time and space complexity. Cruz [8] proposed such a rule using linear updates of values for each video in a dictionary of (video=m, value=$G_u^t(m)$) pairs called a *global list* maintained by the peer. The dictionary for a peer u after t time steps has an entry for each video in the union of videos in storage at other peers contacted by peer u up to time t. The update rule after contacting peer v at time $t+1$ is given by:

$$G_u^{t+1}(m) = \begin{cases} G_u^t(m) + W_u^t(RS_v^t(m)) & m \in S_v^t \\ G_u^t(m) & \text{else} \end{cases}$$

with initial value 0 for any video, where W_u^t is a weighting function that is applied to the relative ranks that the contacted peer has assigned to the videos in its storage. Each global list essentially estimates the popularity of videos stored by contacted peers. Since a more favored video in the master ranking will likely be favored by peers, the global list mechanism will lead to more favored videos having a higher occurrence in the system. Note that the choice of the weighting function plays an important role for estimating the global popularity of videos in the system. We consider the following choices of weighting functions:

Cruz's Top-K Binary Weighting

$$W_v^t(r) = \mathbb{1}(r \leq K).$$

A value in the dictionary for a video is incremented by one each time a contacted peer ranks the video among the top K in its storage.

Positive Linear Weighting

$$W_v^t(r) = \frac{2|S_v(t)| + 1 - 2r}{2|S_v(t)|}.$$

This function decreases from near one to near zero over the range 1 to $|S_v(t)|$. Intuitively, popularity information can be combined more accurately using a graduated weighting function. Videos that are more preferred in the master ranking are also likely to be ranked more highly in peers' personal ranking, which are noisy versions of the master ranking. Thus, higher ranked videos in the master ranking tend to accumulate weights faster. Since C copies of each video are uniformly distributed among peers' starting storages at random, the ranks of the videos stored in any peer with respect to the master ranking are also likely to be distributed uniformly at random. Therefore, the linearity of the weights assigned on each peer's videos takes into account the uniform randomness of the ranks.

Adaptive Linear Weighting

$$W_v^t(r) = \begin{cases} \frac{|S_v(t)|+1-2r}{|S_v(t)|}, & \text{if } |S_u(t)| < S_{max} \\ \frac{2|S_v(t)|+1-2r}{2|S_v(t)|}, & \text{if } |S_u(t)| = S_{max}. \end{cases}$$

This function initially takes the form of a linear function with values ranging from near 1 to near -1. Once the peer's storage reaches its maximum capacity, the adaptive linear weighting function takes the form of a positive linear weighting function. The use of negative weights early in the process is based on the fact that videos are initially distributed uniformly at random. The videos ranked in the bottom half of the starting storage are on average likely to be less desirable than videos not in storage, so a ranking in the lower half of the contacted peer's videos should count negatively compared to a video that has not even been viewed. Later on, the peers tend to have collected preferred videos because less personally preferred videos are eventually removed. The distribution of ranks of the videos stored in each peer with respect to the master ranking becomes denser towards the higher ranked videos, so even a lower half ranking is not necessarily negative compared to no information.

2.3.3 Direct Recommendation Rule

The direct recommendation rule is implemented as follows. In each time step, each peer u pulls the partial ranking $(RS_v^t(m) : m \in S_v(t))$ from its randomly contacted peer v, and then selects the video in it with the highest rank from among those that u has not yet viewed. Ideally, even this limited information exchange allows popular videos (videos that are statistically more likely to be preferred) to be disseminated quickly, with a minimum of complexity.

2.4 Variation: Peers Arriving to Stable System

The pattern of peer arrivals has a strong impact on the happiness of a given peer. The previous section focused on a flash start scenario, in which all peers simultaneously enter the system. As an example of another arrival pattern, essentially the opposite of flash start, we consider a new peer arriving to the system after the system has already stabilized. For this stable system scenario, a new peer arrives with a random uniformly distributed set of videos into a system in which the peers already present have collected their most preferred videos according to their preference functions, $\{m : 1 \leq R_v(m) \leq S_v\}$, where S_v is the storage capacity of peer v. In other words, the peer arrival rate in the stable system scenario is very low.

We propose a counterintuitive hypothesis about systems using the direct recommendation rule: the happiness of a newly arrived peer converges to one more quickly in the flash start scenario than in the stable system scenario. The reason is as follows. Each peer's personal preference function is a noisy version of the master ranking generated by the PL ranking model, so the personal preferences of all the peers are statistically correlated. In the stable system, the set of videos ranked most highly by each peer are available. In the flash start system, higher ranked videos are downloaded iteratively into each peer's storage. While these are the videos ranked most highly by the contacted peer at the time of contact, they may not be among the videos with the highest ranking in the contacted peers personal ranking. Therefore, it might seem that the availability of other peers' personally preferred videos would be beneficial.

Although in flash start every peer acquires its favorite videos slowly, we argue that the happiness of each peer converges to one faster because of the effect of belief propagation. Because of the PL model, a video that is ranked higher in the master ranking is also more likely be ranked higher by other peers by statistical correlation. Since the highest ranked video in each peer's storage is recommended and downloaded in each time step, a video that is ranked higher in the master ranking will propagate more frequently and become more popular than any lower ranked video in the master ranking. Then the subtle difference between flash start and stable system is that popular videos are downloaded in flash start and contacted peer's personally preferred videos are downloaded in stable system. Video popularity has a better statistical representation of the master ranking in the flash start scenario than in the personal preferences of peers. Therefore, under direct recommendation, happiness converges faster to one in flash start than in stable system.

2.5 Performance for Ranking Model

To compare the three recommendation rules of Section 2.3 along with the upper bound of Section 2.2 for application to the ranking model, we simulated them for the system parameters used in [8]: 1000 peers, 1000 videos, storage capacity per peer 100 videos, each peer's personal preference is constructed using the PL distribution with Zipf weights with parameter $\alpha = 2.25$, and each peer is initially seeded with 30 videos selected uniformly at random. For the purposes of performance evaluation, as in [8], the happiness $H_n(t)$ of a peer n at time t is defined to be the fraction of the videos the peer has from among the 50 top videos in the peer's personal ranking of videos. The system happiness at any time is the average over all peers of the peer happiness. Each curve in each plot is the system happiness vs. time for one simulation run. For the direct recommendation rule, additional simulations are presented to evaluate the robustness of the performance.

2.5.1 Performance of MLE Recommendation Rule and Upper Bounds

Figure 2.1 shows the happiness vs. time for (i) the MLE rule, (ii) the global list recommendation rule with the adaptive linear choice of W, (iii) the upper bound of Section 2.2, and (iv) the trivial upper bound of Section 2.2 (giving rise to the straight line of slope 1/50). Both the MLE recommendation rule and the global list recommendation rule give performance very close to the upper bound, with the MLE having a slightly better performance than the global list recommendation rule.

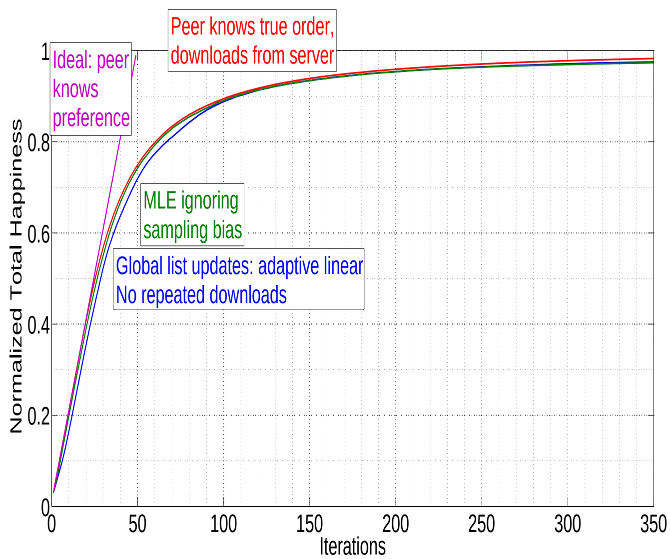

Figure 2.1: Video content collection performance with partial ranks and upper bounds in a system of 1000 peers, 1000 videos, 100 storage size, and $k = 50$

2.5.2 Performance of Global List Recommendation Rule

We examine the global list recommendation rule, with a focus on the impact of the choice of weight function W discussed in Section 2.3.2. Figure 2.1 shows the performance curve of global list recommendation rule with adaptive linear weighting function. Figure 2.2 indicates the happiness vs. time when the weight function W used in the global list recommendation rule is Cruz's top-K weighting function with $K = 50$. Two curves are shown. For the upper curve, peers do not download the same video twice. The performance is sluggish for small t, which is to be expected; if a peer has at most $K = 50$ videos it will rate all of them in its top 50.[1]

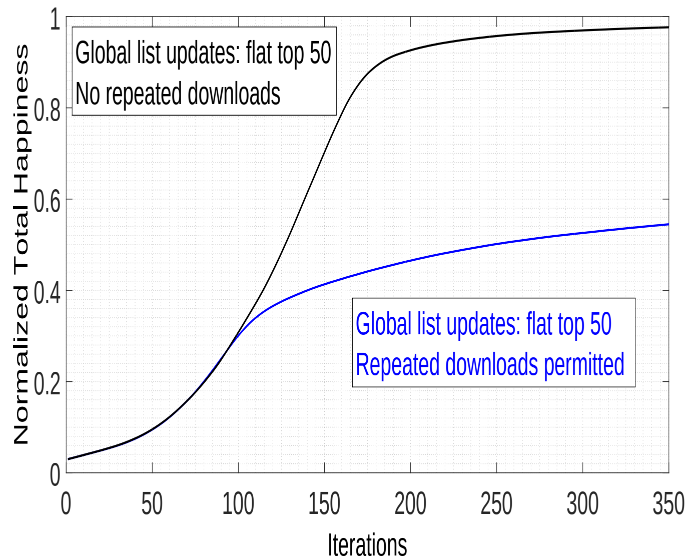

Figure 2.2: Video content collection performance with Cruz's binary weighting function in a system of 1000 peers, 1000 videos, 100 storage size, and $k = 50$

The performance curves of the global list recommendation rule for all three choices of weighting function W described in Section 2.3.2 are shown in Figure 2.3. Both the positive linear W and the adaptive linear W perform substantially better than the top-50 binary W, with adaptive linear doing slightly better than positive linear.

2.5.3 Direct Recommendation - Robustness of Performance

In the simulations we have found that the direct recommendation rule performs nearly as well as the more complex MLE recommendation rule or the global list recommendation rule with positive linear or adaptive linear W.

[1] Simulation of the same system appears much better in Cruz's paper, possibly due to leaking of master ranking information to peers during tie breaking.

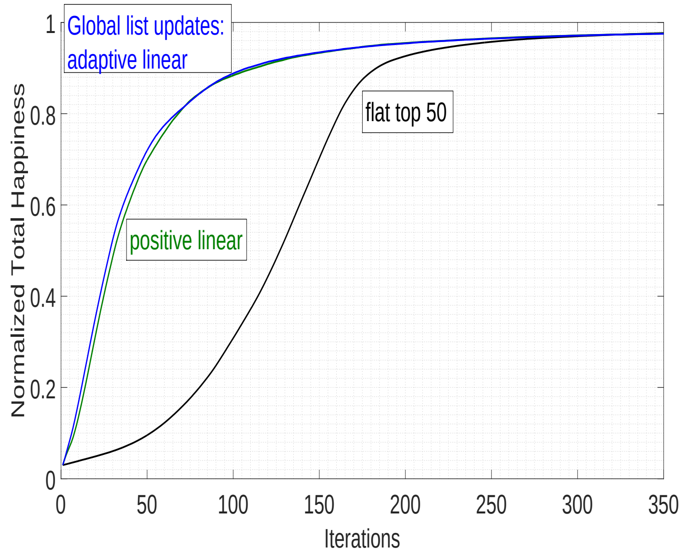

Figure 2.3: Video content collection performance with linear weighting functions in a system of 1000 peers, 1000 videos, and 100 storage size

The performance curves are shown in Figure 2.4. In addition, the robustness of the direct recommendation rule is explored under a variety of assumptions.

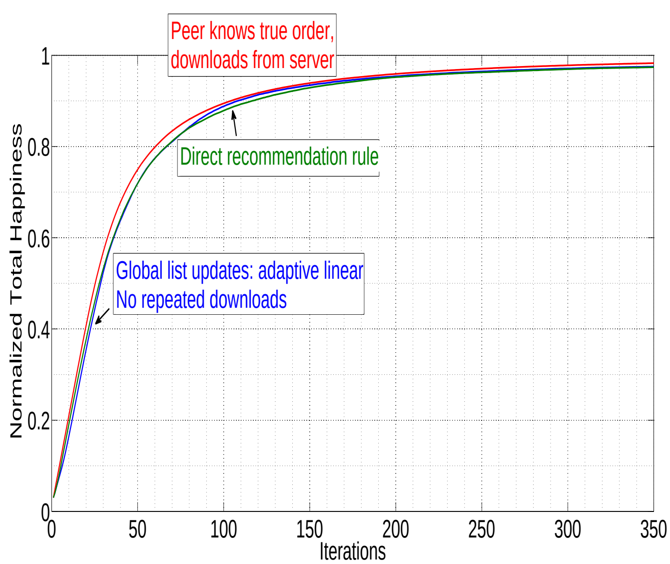

Figure 2.4: Video content collection performance with direct recommendation in a system of 1000 peers, 1000 videos, 100 storage size, and $k = 50$

Varying α

Recall each peer's personal preference is constructed using the PL ranking model with Zipf weights $w_m \propto m^{-\alpha}$. The similarity among peers is increasing in α. At the extreme value $\alpha = 0$, the peers have independent preferences, and at the extreme $\alpha \to \infty$ the peers have identical preferences. Figure 2.5 shows the upper bound and the performance of the direct recommendation rule for $\alpha \in \{2.25, 1.25, 0.8, 0.6, 0.2, 0\}$. The performance strongly depends on α. However, it appears that the performance gap between the direct recommendation rule and the upper bound is small over the complete range of α, with the largest gap for α in the range 0.6 to 0.8. This illustrates the robustness of the direct recommendation rule under different personal preference distributions.

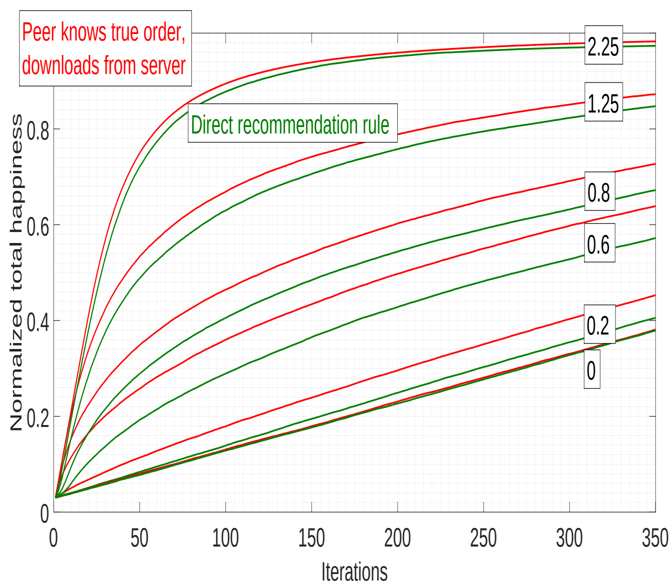

Figure 2.5: Video content collection performance with direct recommendation in a system of 1000 peers, 1000 videos, and 100 storage size, for several different Zipf parameter values

Varying Initial Video Availability

Recall each peer is initially seeded with 30 videos selected uniformly at random. To check the performance of the direct recommendation rule at different initial video availabilities, we have performed simulations with fewer initial video availabilities at each peer. Figure 2.6 shows the performance when each peer is initially seeded with 10, 5 or 1 videos selected uniformly at random, respectively. It appears that the performance gaps of the direct recommendation rule over different initial video availabilities are small when each peer is initially seeded at least 5 videos. If each peer is seeded with only one randomly selected video and $M = N$, then the probability a particular video is initially present in the system is $1-(1-\frac{1}{N})^N \approx 1-e^{-1} \approx 62\%$. This limits the normalized total system happiness to about 62%. Figure 2.6 thus shows near optimal performance for the direct recommendation rule even for one initial available video per peer. This illustrates the robustness of the direct recommendation rule under more restrictive initial video availabilities.

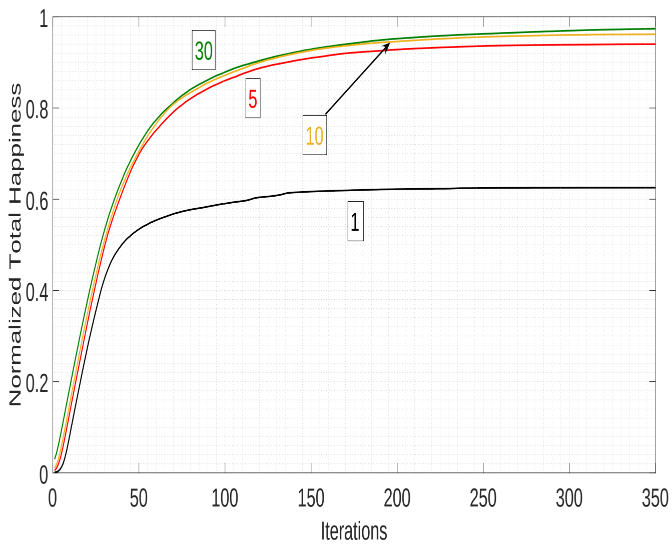

Figure 2.6: Video content collection performance with direct recommendation in a system of 1000 peers, 1000 videos, and 100 storage sizes, for several different initial video availabilities

Varying Storage Size

Recall each peer has a storage capacity of 100 videos. To check the performance of the direct recommendation rule with different storage sizes, we have performed simulations with smaller storage sizes at each peer. Figure 2.7 shows the performance when each peer has a storage capacity of 75, 50 or 30 videos, respectively. It appears that the performance gaps of the direct recommendation rule over different storage sizes are small when storage sizes are at least 50 videos, which is the threshold used in the happiness function. If each peer has a storage capacity of 30 videos, which is the number of videos initially available to each peer, then the normalized total system happiness is limited to at most $\frac{30}{50} = 60\%$. Figure 2.7 thus shows near optimal performance for the direct recommendation rule even for storage capacity of 30 videos per peer. This illustrates the robustness of the direct recommendation rule under more restrictive storage sizes.

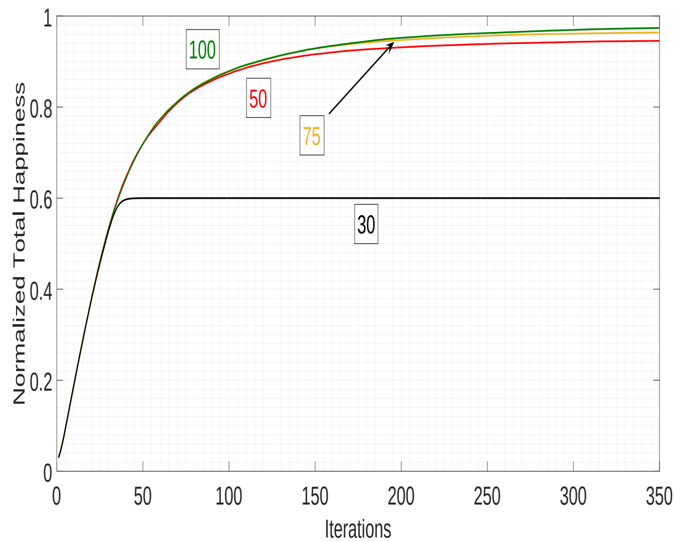

Figure 2.7: Video content collection performance with direct recommendation in a system of 1000 peers, 1000 videos, and various storage sizes

Variation: Peers Arriving to Stable System

To check the performance of the direct recommendation rule at different peer arrival rates, we have performed simulations on a peer arriving under two system scenarios: flash start and stable system. The corresponding performance curves of the direct recommendation rule under the two system states are shown in Figure 2.8.

The simulations give evidence in favor of our hypothesis that flash start helps peers acquire their favored videos better than a stable system using the direct recommendation rule due to the belief propagation effect (see Section 2.4), although the differences are small. This illustrates the robustness of the

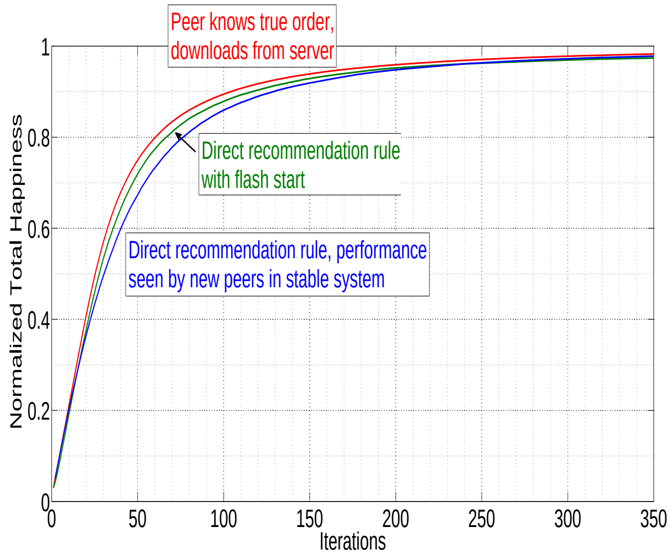

Figure 2.8: Effect of belief propagation for video content collection with direct recommendation in a system of 1000 peers, 1000 videos, 100 storage size, and $k = 50$

direct recommendation rule under the two extreme peer arrival patterns.

Varying Happiness Function

Recall the happiness function $H_n(t)$ of a peer n at time t used for the simulations in this section is the fraction of the videos the peer has from among the 50 top videos in the peer's personal ranking of videos, which is a step function with threshold 50. Any happiness function can be generated as a weighted sum of step functions with different thresholds. Thus, to check for robustness with respect to the choice of happiness function, it suffices to check the performance of the direct recommendation rule under step happiness functions with different thresholds. Figure 2.9 shows the upper bound and the performance of the direct recommendation rule under step functions with thresholds $\{25, 50, 75, 100, 150, 200, 300, 400, 700, 1000\}$. For thresholds greater than 100, the maximum normalized total happiness is limited by the selected storage size of 100. It appears that the performance gap between the direct recommendation rule and the upper bound is small over the complete range of thresholds. This illustrates the robustness of the direct recommendation rule under a broad range of happiness functions.

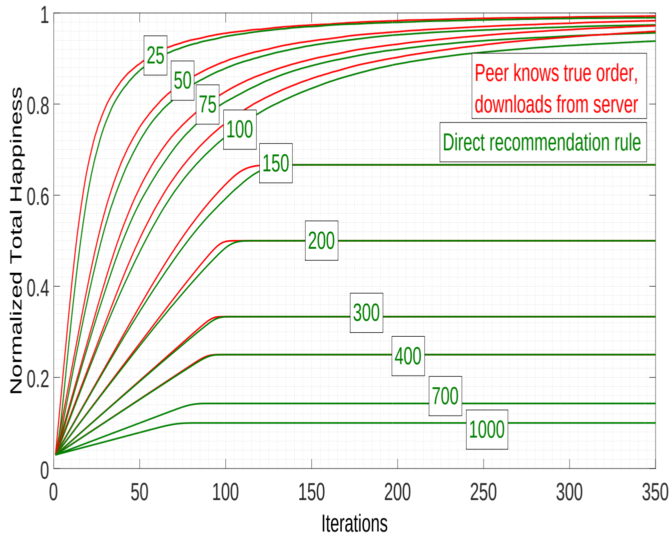

Figure 2.9: Video content collection performance with direct recommendation in a system of 1000 peers, 1000 videos, and various thresholds in happiness step functions

2.6 Summary of Results

In the single-cluster regime with homogeneous population and PL ranking model, we analyzed the distributed recommendation system that jointly performs content collection and rank aggregation. We found a performance upper bound using stochastic comparison. Using the intuition obtained from the performance upper bound, three recommendation rules of different complexity are proposed. We proposed a deluxe recommendation rule that estimates the master ranking using MLE. The deluxe recommendation rule used the EM algorithm on partial rankings and was found to be near optimal in the simulations. We reevaluated Cruz's recommendation rule and obtained insights in the global list recommendation rule and the weight functions. With the intuition, we tried to optimize this approach and proposed more efficient variations of the rule using different weighting functions to produce an aggregate of partial rankings. Lastly, we proposed a simple greedy recommendation rule called direct recommendation and found that its performance is also near optimal in the simulations.

A main conclusion is that distributed content collection and rank aggregation is not only feasible, but the proposed direct recommendation rule works remarkably well over a broad range of system parameters including strength of correlation among peers, initial number of videos available, storage sizes, peer arrival patterns, and choice of happiness functions. This conclusion is illustrated by the simulations described in Section 2.5.3.

2.7 Derivation of EM Algorithm for Estimating the Master Ranking

P. S. Efraimidis made the connection between WSNR and the collection of independent exponential random variables [26]. Specifically, given a probability vector $\overline{\theta} = (\theta_1, \theta_2, ..., \theta_n)$, let $\overline{X} = \{X_1, X_2, ..., X_n\}$ be a vector of independent exponential random variables with the given distribution rates $X_i \sim Exp(\theta_i)$ for $1 \le i \le n$. The set of the exponential random variables produces an ordered set of exponential jumps such that the orders follow the same probability distribution as the order produced by WSNR. For example, let $n = 3$ and suppose the order of exponential jumps is $\{X_1, X_3, X_2\}$, then X_1 occurs before X_2 and X_3 with probability $\frac{\theta_1}{\theta_1+\theta_2+\theta_3}$, and by the memoryless property of exponential random variable, X_3 occurs before X_2 with probability $\frac{\theta_3}{\theta_2+\theta_3}$.

Suppose we ignore the fact that θ in PL model is a permutation of the Zipf distribution. Then, we have a convex problem to find MLE of θ. The exponential representation will simplify the estimation method, i.e., the MM algorithm can be reduced to the EM algorithm for our model.

For completeness, we derive the EM algorithm. Let $\theta = \{\theta_1, \theta_2, ..., \theta_{|M|}\}$ be the parameter to be estimated. The estimated parameters are the scaled Zipf parameters. Let $X = \{\overline{X}_{C_u(1),1}, \overline{X}_{C_u(2),2}, ..., \overline{X}_{C_u(t),t}\}$ be the complete data, which is the vector of exponential variables sets with Zipf parameters as the rates, i.e. $\overline{X}_{C_u(s),s} = \{x_{(C_u(s),m,s)} | m \in |M|\}$. Let $Y = \{PR^1_{C_u(1)}, PR^2_{C_u(2)}, ..., RV^t_{C_u(t)}\}$ be the observed direct ranking vectors. Let $PF^t_{C_u(t)}(r)$, $1 \le r \le V_{C_u(t)}(t)$, be the partial preference function, which is the inverse of $RV^t_{C_u(t)}$. Then, the conditional probability of the complete data given θ is $\mathbb{P}_{cd}(X|\theta)$. We derive the expectation step and the maximization step below.

$$\begin{aligned}
\mathbb{P}_{cd}(X = x|\theta) &= \prod_{s=1}^{t} \mathbb{P}(\overline{X}_{C_u(s)} = \overline{x}_{C_u(s)}|\overline{\theta}) \\
&= \prod_{s=1}^{t} \prod_{m \in V_{C_u(s)}(s)} \theta_m e^{-\theta_m x_{(C_u(s),m)}} && (2.3) \\
log\mathbb{P}_{cd}(x|\theta) &= \sum_{s=1}^{t} \sum_{m \in V_{C_u(s)}(s)} \log\theta_m - \theta_m x_{(C_u(s),m)}. && (2.4)
\end{aligned}$$

The expectation step is

$$\begin{aligned}
Q(\theta|\theta^{(k)}) &= E[log\mathbb{P}_{cd}(x|\theta)|y, \theta^{(k)}] \\
&= -\sum_{s=1}^{t} \sum_{m \in V_{C_u(s)}(s)} \theta_m E[x_{(C_u(s),m)}|PR^s_{C_u(s)}, \theta^{(k)}] + \log\theta_m. && (2.5)
\end{aligned}$$

Taking the partial derivative

$$\frac{\partial Q(\theta|\theta^{(k)})}{\partial\theta_m} = -\sum_{s:m \in V_{C_u(s)}(s)} E[x_{(C_u(s),m)}|PR^s_{C_u(s)}, \theta^{(k)}] + \frac{1}{\theta_m}. \quad (2.6)$$

Taking the partial derivative $\frac{\partial Q(\theta|\theta^{(k)})}{\partial\theta_m} = 0$, the maximization step is

$$\theta_m^{(k+1)} = \frac{|\{s : m \in V_{C_u(s)}(s)\}|}{\sum_{s:m \in V_{C_u(s)}} E[x_{(C_u(s),m)}|PR^s_{C_u(s)}, \theta^{(k)}]}, \quad (2.7)$$

where $E[x_{(C_u(s),m)}|PR^s_{C_u(s)}, \theta^{(k)}] = \sum_{j=1}^{PR^s_{C_u(s)}(m)} \left(\frac{1}{\sum_{m \in V_{C_u(s)}(s)} \theta_m^{(k)} - \sum_{l=1}^{j-1} \theta^{(k)}_{PF^s_{C_u(s)}(l)}} \right)$.

CHAPTER 3

SCORING SYSTEM AND LARGE SYSTEM SCALING

An alternative to classifying videos by ranking them is for peers to classify videos by assigning scores. We shall consider a model for correlated score assignments by peers similar to the PL model for correlated rankings by peers. At least for the direct recommendation rule, this provides some analytical tractability, and, as shown in Section 3.5, the analysis can be applied back to the PL ranking model.

3.1 Independent Crossover Channel Model

For the independent crossover (IC) channel model, there are L possible scores, $[L] = \{1, ..., L\}$, for each video. Similar to the PL model, given a master score vector, peers' preferences are constructed so that they are conditionally independent given the master score vector. To be consistent with the ranking order in the previous sections, we let lower numbered scores represent more preferred videos. Let M denote the number of videos and let $G_1, \ldots, G_L$ be disjoint subsets of $[M]$, with union equal to $[M]$, and call the videos indexed by G_ℓ *type* ℓ *videos*. This partition is equivalent to giving a master score vector, where the video types are master scores. Let $m_\ell = |G_\ell|$ for $\ell \in [L]$.

The IC channel model for L, $G_1, \ldots, G_L$, and an $L \times L$ stochastic matrix W is defined as follows. For any video in G_ℓ, a peer assigns personal score ℓ' to the video with probability $W_{\ell\ell'}$. The scores assigned by all peers to all videos are assumed to be independent, given the types of the videos. Each peer n for $n \in [N]$ has an intrinsic personal scores of videos, $i_n : [M] \to [L]$. We can and do take the viewpoint that a peer decides what score to assign to a video upon downloading the video.

The problem of distributed content collection and score aggregation among N peers can be formulated for scores and the IC model just as it was for rankings and the PL model. In particular, the happiness of a peer at time t can be defined in terms of a happiness function, based on the scores of the videos obtained by the peer up to time t. The direct recommendation rule carries over with no change; when a peer contacts another peer, the video downloaded is uniformly randomly selected from among the highest scored (by the contacted peer) videos that are available at the contacted peer and not yet possessed by the contacting peer. *For analytical tractability throughout the remainder of this section, we consider the* IC *model used under the direct recommendation rule.* We also show how the method can be applied to give an approximate analysis for the PL ranking model used under the direct recommendation rule.

If A, B are finite multisets of $\mathbb{R}$, $A \succ B$ (A is better than B) indicates that $|A| \geq |B|$ and $a_{[i]} \leq b_{[i]}$ for $1 \leq i \leq |B|$, where $a_{[1]} \leq \ldots \leq a_{[|A|]}$ denotes the ordered elements of A and $b_{[1]} \leq \ldots \leq b_{[|B|]}$ denotes the ordered elements of B. The happiness function of a peer at time t is defined as $H_n(t) = f(\{i_n(m) : m \in S_n(t)\})$, where $f : 2^{[M]} \to \mathbb{R}$, is assumed to be nondecreasing in the $\succ$ order, and $2^{[M]}$ denotes the set of subsets of $[M]$.

3.2 An Upper Bound on System Performance

To obtain an upper bound on the happiness of a peer, no matter what recommendation rule is used, consider an idealized system in which the peer has access to a server that can provide any video, and for which a genie reveals extra information. If the genie revealed to the peer the peer's own personal scores of the videos, $(i(m) : m \in [M])$, then the obviously optimal rule of the peer would be to download videos in the order of increasing numerical personal scores. This provides a rather trivial upper bound on the happiness of a peer vs. time. Our focus is on a tighter upper bound, derived by considering the case in which the peer has access to a server, and the genie reveals only the master scores to the peer. Since the preferences of other peers in the system are conditionally independent given the master scores, the scores of other peers provide no additional clues to the peer about its own preferences. The following theorem shows the peer's optimal recommendation rule is to download the videos in the order of increasing master score (e.g. increasing index m).

Theorem 3.2.1. *Let $f : 2^{[M]} \to [0,1]$ be nondecreasing in the happiness order $\prec$. Let $(i(m) : m \in [M])$ denote a random scoring vector generated by the* IC *model for some $G_1, ..., G_L$ and W. Suppose the rows of W are increasing in the usual stochastic order sense. If $A, B \subset [M]$ such that $A \succ B$, then $\mathbb{E}\left[f(\{i(m) : m \in A\})\right] \geq \mathbb{E}\left[f(\{i(m) : m \in B\})\right]$.*

Proof of Theorem 3.2.1. Let $a_{[1]}, ..., a_{[|A|]}$ and $b_{[1]}, ..., b_{[B]}$ denote the ordered elements of A and B respectively. By assumption, $|A| \geq |B|$ and $a_{[m]} \leq b_{[m]}$ for $1 \leq m \leq |B|$. Since the rows of W are nondecreasing in the stochastic order sense, the scores $(i(m) : m \in A)$ and $(i(m) : m \in B)$ can be coupled as follows. There exists $(i'(m) : m \in A)$ and $(i''(m) : m \in B)$ on one probability space such that (i) $(i'(m) : m \in A) \stackrel{d.}{=} (i(m) : m \in A)$, (ii) $(i''(m) : m \in B) \stackrel{d.}{=} (i(m) : m \in B)$, and (iii) $\mathbb{P}\{(i'(m) : m \in A) \succ (i''(m) : m \in B)\} = 1$. Thus, $\mathbb{P}\{f\{i'(m) : m \in A\}\} \leq f\{i''(m) : m \in B\}\} = 1$, so $E[f\{i'(m) : m \in A\}] \leq E[f\{i''(m) : m \in B\}]$. $\square$

In words, if a set A of videos is better than a set B of videos in the master order, then A is stochastically better than B for a peer. The bound suggests that performance in the original system would be nearly optimal if (1) the peers could quickly and accurately infer the master order by sharing preference information, and (2) the peer-to-peer content distribution mechanism could provide requested videos nearly as well as a centralized server.

3.3 The Mean Field Limit ($N \to \infty$)

The state of the system at a given time consists of the states of all N peers. Each peer has $(L+1)^M$ possible states, which we refer to as *detailed states* of the peers. The detailed state of a peer can be written as $i = (i(m) : m \in [M])$ where $i(m) \in \{0, \ldots, L\}$ indicates the score the peer assigns to video m, with value zero denoting that the peer does not yet have the video. In a system of N peers, let $X_n^N(t)$ represent the state of peer n at time t. By definition, the empirical distribution of peers in the system at time t, denoted by $M^N(t)$, assigns probability $M_i^N(t) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{\{X_n^N(t)=i\}}$ to i for each possible detailed state i of a peer. Then, each peer's state transition probabilities given the state of the system are denoted by $K_{ij}^N(\vec{r}) \triangleq \mathbb{P}\{X_n^N(t+1) = j|\vec{M}^N(t) =$

$\vec{r}, X_n^N(t) = i\}$, where $\vec{r}$ is a probability vector indexed by detailed states with coordinates being a multiple of $\frac{1}{N}$.

The number of possible system states, $(L+1)^{MN}$, grows exponentially with N. Fortunately, because peers contact each other uniformly at random, the ordering of peers is not relevant. By this exchangeability among peers, the detailed state of the system at a given time can be represented by the sequence of empirical distributions $(\vec{M}^N(t) : t \geq 0)$, which forms a Markov sequence. If we let $N \to \infty$, we can apply mean field theory to the system, implying that the empirical distribution becomes deterministic by the following theorem.

Theorem 3.3.1. *[27] Let $(X_n^N(t) : t \geq 0, \forall n \in N)$ be a sequence of states for objects in a system such that the sequence of the empirical distributions of objects in the system $(\vec{M}_i^N(t) : t \geq 0)$ is a Markov sequence and the state transitions for individual objects at time $t+1$ are conditionally independent given the current state at time t. Suppose the transition probabilities have the form $\mathbb{P}\{X_n^N(t+1) = j | \vec{M}^N(t) = \vec{r}, X_n^N(t) = i\} = K_{ij}^N(\vec{r})$. Assume for all i, j, for $N \to \infty$, $K_{ij}^N(\vec{r})$ converges uniformly in $\vec{r}$ to some $K_{ij}(\vec{r})$, which is a continuous function of $\vec{r}$. Also assume that the initial empirical distribution $\vec{M}^N(0)$ converges almost surely to a deterministic limit $\vec{\mu}(0)$. Define $\vec{\mu}(t)$ iteratively by its initial value $\vec{\mu}(0)$ and for $t \geq 0$:*

$$\vec{\mu}_j(t+1) = \sum_i \vec{\mu}_i(t) K_{ij}(\vec{\mu}(t)). \tag{3.1}$$

Then for any fixed $t \geq 0$, almost surely, $\lim_{N\to\infty} \vec{M}^N(t) = \vec{\mu}(t)$.

Theorem 3.3.1 directly applies to the IC model with the direct recommendation rule. For that application, $K_{ij}^N(\vec{r})$ does not depend on N when peers are allowed to contact themselves, so $K_{ij}^N(\vec{r}) \equiv K_{ij}(\vec{r})$. Since $K_{ij}(\vec{r})$ is a linear combination of the coordinates of $\vec{r}$, it is a continuous function of $\vec{r}$. $\vec{M}^N(0)$ is the empirical distribution of N i.i.d. random variables with distribution not dependent on N; we assume the same fraction of videos are chosen uniformly at random to be initially stored in each peer. Therefore, $\vec{M}^N(0) = \lim_{N\to\infty} \vec{M}^N(0)$ a.s., where $\vec{M}(0)$ is the initial state distribution for a peer. Theorem 3.3.1 gives rise to a mean field model, in which there is a single *tagged peer* with a Markov evolution, such that for a time step from t to $t+1$, the tagged peer interacts with a *contacted peer* selected using the probability vector $\vec{\mu}(t)$, independently of the past history of the tagged peer. The vector $\vec{\mu}(t)$ is also the distribution of the tagged peer at time t.

While the evolution of the empirical distribution of the detailed states is deterministic in the limit $N \to \infty$, the number of possible states for a single peer, $(L+1)^M$, is still so large that solving the update equation (3.1) is not computationally feasible for realistic choices of L and M. We next describe how the state space can be reduced substantially by exploiting symmetry. In the original system the initial distribution and the dynamics are invariant with respect to reordering of the videos within each group G_ℓ, and that invariance is inherited by the mean field limit. Specifically, if i is a possible detailed state of a peer, let $Z_{\ell\ell'}(i)$ be the number of videos the peer has with master score ℓ and personal score ℓ'. Define two states i and i' to be equivalent if $Z(i) \equiv Z(i')$. By the symmetry noted above, it follows that $\vec{\mu}_i(t) = \vec{\mu}_{i'}(t)$. We refer to $(Z_{\ell\ell'}(i) : \ell, \ell' \in [L])$ as the *reduced state* of a peer. A peer has $\prod_{\ell=1}^{L} \binom{L+m_\ell}{m_\ell}$ possible reduced states. For the mean field model, it suffices to track the probabilities of reduced states because the probability of a detailed state can be recovered by dividing the probability of its corresponding reduced state by the number of equivalent detailed states. The sequence of reduced states of the tagged peer in the mean field model continues to be a Markov process.

Another observation can be used to further reduce the number of states considered. The determination of which video the tagged peer downloads from a contacted peer does not involve how the tagged peer has rated its own videos. Therefore, if $(Z_\ell : \ell \in [L])$ denotes how many videos the tagged peer has from each of the master groups G_ℓ at some time t, the scores assigned to those videos by the tagged peer are conditionally independent, with the scores of videos in G_ℓ being assigned using the ℓ^{th} row of W. For determination of the state of the tagged peer at time $t+1$, it does matter how the contacted peer has rated the videos it has. But the distribution of the state of the contacted peer is the same as that of the tagged peer, so the scores the contacted peer has assigned to the videos it has from each master score group G_ℓ are also conditionally independent and generated according to the ℓ^{th} row of W. Equivalently, it is as if the contacted peer scores each of the videos it has from each master group G_ℓ at the time of contact by the tagged peer. Therefore, the mean field probability $\vec{\mu}(t)$ can be computed by only keeping track of the distribution of $(Z_\ell : \ell \in [L])$ for the tagged peer. Given $(Z_\ell : \ell \in [L])$ for a peer at some time, the conditional distribution of detailed state is as follows. Independently, for each ℓ, the conditional distribution of the m_ℓ coordinates $(i(m) : m \in G_\ell)$ is that there are $m_\ell - Z_\ell$ zeros selected uniformly at random from among the m_ℓ possible locations, and Z_ℓ nonzero scores, each selected independently according to the ℓ^{th} row of W. We refer to $(Z_\ell : \ell \in [L])$ as the *more reduced state* of a peer. There are $\prod_{\ell=1}^{L}(m_\ell + 1)$ possible more reduced states.

In closing this section, we describe the mean field model for the more reduced states in more detail. It has the following parameters: positive integers $M, L, m_1, \ldots, m_L$, such that $M = m_1 + \ldots + m_L$, an $L \times L$ crossover matrix W, and a constant $c > 0$ denoting the fraction of videos initially assigned to each peer. We switch to using $\mu_t^{(M)}$ for the mean field limit distribution at time t, to denote the dependence on M, in anticipation of the next section. The state space for the model is $S^{(M)} = \left\{ z \in \mathbb{Z}_+^L : 0 \leq z_\ell \leq m_\ell \right\}$. The dynamical aspect of the mean field model is described by a function $\Phi^{(M)} : S^{(M)} \times S^{(M)} \to \Sigma^{(L)}$, where $\Sigma^{(L)} = \{p \in \mathbb{R}_+^L : \sum_\ell p_\ell \leq 1\}$. The interpretation is that if the state of the tagged peer is z and the randomly contacted peer has state z', then $\Phi_\ell^{(M)}(z, z')$ is the probability that the tagged peer downloads a type ℓ video from the contacted peer. The detailed specification of $\Phi^{(M)}(z, z')$ is a bit complicated but can be briefly explained as an algorithm, as follows. Given z and z', first, for each ℓ, generate a random variable representing the number of type ℓ videos eligible for download, where eligible for download means three conditions are satisfied: (i) the contacted peer has the video, (ii) the tagged peer does not have the video, and (iii) the contacted peer classifies the video as type 1 (true with probability $W_{\ell 1}$). If at least one video is eligible for download, one such video is selected for download uniformly at random from among those that are eligible. If no such videos are eligible, repeat steps (i)-(iii) seeking a video of type 2 to download, and so on. If the contacted peer has no video that the tagged peer does not have, no video is downloaded.

The mean field model determines a sequence $(\mu_t^{(M)} : t \geq 0)$ of probability distributions (represented as vectors) on $S^{(M)}$. These distributions are determined recursively as follows. The initial distribution, $\mu_0^{(M)}$, corresponds to selecting Mc videos uniformly at random from $[M]$ and recording the number of each type selected. Given $\mu_t^{(M)}$ for some $t \geq 0$, states Z and Z', corresponding to a tagged peer and a contacted peer, are independently generated with distribution $\mu_t^{(M)}$, and then the video downloaded by the tagged peer is determined using the distribution $\Phi^{(M)}(Z, Z')$. If the video is type ℓ, the new state of the tagged peer Z is modified by increasing the ℓ^{th} coordinate by one. Then, $\mu_{t+1}^{(M)}$ is the probability distribution of the new state of the tagged peer.

Once the sequence of distributions $(\mu_t^{(M)} : t \geq 0)$ has been calculated, we can define a Markov process modeling the entire time history, $(Z_t : t \geq 0) = (Z_{t,\ell} : t \geq 0, \ell \in [L])$, of a tagged peer as follows. The initial state Z_0 has distribution $\mu_0^{(M)}$, and, given Z_t, the distribution of what type video is downloaded to get Z_{t+1} is given by $\Phi^{(M,1,t)}(Z_t)$, where

$$\Phi^{(M,1,t)}(z) \triangleq \sum_{z' \in S^{(M)}} \Phi^{(M)}(z, z') \mu_{t,z'}^{(M)}.$$

Like $\Phi^{(M)}$, $\Phi^{(M,1,t)}$ takes values in $\Sigma^{(L)}$. The "1" and "t" in the notation $\Phi^{(M,1,t)}(z)$ indicate that one argument of $\Phi^{(M)}(z, z')$ remains after z', corresponding to the state of the contacted peer, is averaged out using the distribution $\mu_t^{(M)}$, which depends on t. Induction on t shows that Z_t has distribution $\mu_t^{(M)}$ for each t. This completes our description of the mean field model.

3.4 Fluid Limit of the Mean Field Model ($M \to \infty$)

Although the number of possible more reduced states is much smaller than the number of detailed states, it is still rather large and, furthermore, exact computation of the state transition matrix for the more reduced states essentially requires expanding to the reduced states and is computationally expensive. For example, for 1000 videos, binary scores, and 50 videos with the higher score, there are about 5×10^4 more reduced states, so the time dependent transition probability matrix for the state of the tagged peer has $(5 \times 10^4)^2$ entries. To reduce the complexity further we establish a fluid limit of the mean field model as the number of videos converges to infinity. The limit takes advantage of the fact, due to the law of large numbers, that the distribution of the more reduced state of the tagged peer tends to concentrate around its mean. This entails the limit of a limit, because the mean field model itself arises as the number of peers converges to infinity.

Consider a sequence of mean field models as $M \to \infty$ with L, c, and an

$L \times L$ crossover probability matrix W fixed. Also, let $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_L)$ be a fixed probability vector with positive coordinates, and let $(m_1, \ldots, m_L)$ depend on L in such a way that $m_\ell = \rho_\ell M$ for $\ell \in [L]$. To avoid trivial complications, we assume Mc and m_ℓ for each ℓ are integer valued; this can be done, for example, by assuming c and the coordinates of $\boldsymbol{\rho}$ are multiples of 0.01 and M is a multiple of 100. Let

$$\varphi_\ell(x, y) \triangleq \frac{(1 - x_\ell/\rho_\ell)y_\ell W_{\ell 1}}{\sum_{\ell'}(1 - x_{\ell'}/\rho_{\ell'})y_{\ell'}W_{\ell' 1}}$$

and let $(b_\tau : 0 \leq \tau \leq (1-c))$ denote the solution to $\dot{b}_\tau = \varphi(b_\tau, b_\tau)$ with initial state $b_0 = c\boldsymbol{\rho}$. The following theorem is proved in Section 3.8.1.

Theorem 3.4.1. *Suppose the first column of W has strictly positive entries. As $M \to \infty$, $\sup_{0 \leq t \leq M(1-c)} \|\frac{Z_t}{M} - b_{t/M}\| \to 0$, where the convergence is in probability.*

The idea of the proof of Theorem 3.4.1 is as follows. Classical techniques from the theory of differential equation limits of Markov processes (see for example, [28] and the references therein) are applied to show that as $M \to \infty$, the scaled stochastic trajectory of Z concentrates along a deterministic path. It follows that the peers contacted by the tagged peer also have states near the same deterministic trajectory. The deterministic trajectory is identified by drift analysis, which in the limit $m \to \infty$ after scaling in time and space gives rise to the integral equation that is equivalent to the differential equation determining b.

We next discuss the approximation suggested by Theorem 3.4.1. The theorem shows that for large M, Z_t is well approximated by $z_t \triangleq M b_{t/M}$. Expressing the differential equation for b in terms of z yields

$$\dot{z}_{t,\ell} = \frac{(1 - \frac{z_{t,\ell}}{m_\ell})z_{t,\ell}W_{\ell 1}}{\sum_{\ell'}(1 - \frac{z_{t,\ell'}}{m_{\ell'}})z_{t,\ell'}W_{\ell' 1}}, \qquad z_0 = \boldsymbol{\rho} Mc. \tag{3.2}$$

The righthand side of (3.2) has a natural interpretation. The numerator represents the approximate number of videos from master group ℓ that are eligible for transfer from the contacted peer, and the whole righthand side represents the fraction of eligible videos that are from master group ℓ. All these videos are among those assigned score one by the contacted peer. By differentiation we can verify that the solution to the differential equation can be represented in parametric form by:

$$z_{t,\ell} = \frac{m_\ell}{(\frac{1}{c}-1)e^{-\gamma_t W_{l1}}+1} \tag{3.3}$$

$$\sum_\ell z_{t,\ell} = Mc + t. \tag{3.4}$$

By letting γ vary over $[0,\infty]$ we can numerically sweep out the trajectory (z_t, t) over $0 \le t \le M(1-c)$. Moreover, the number of videos a peer has at time t from master group ℓ that are given a personal score ℓ' by the peer is well approximated by $z_{t,\ell}W_{\ell\ell'}$, so we can also evaluate the happiness of a peer vs. time for M large.

3.5 Analysis of the Plackett-Luce Model Using IC Channel Model

Our analysis of the IC model is applied in this section to analyze the PL model under the direct recommendation rule. The PL model is similar to the IC model in the sense that it has an induced crossover probability matrix, namely, $\widetilde{W}_{ij}^{(M)} \triangleq \mathbb{P}\{R_n(i) = j\}$ for all $i,j \in [M]$. However, it is different because the variables $R_n(i)$ are not independent. For example, in any permutation, no two items can have the same rank. Theorem 3.4.1, justifying the fluid limit, assumes that the number of possible scores, L, is fixed, as $M \to \infty$. However, if we ignore the two assumptions, namely, independent crossovers and L fixed, then we are naturally led to apply the mean field limit where we use crossover matrix $\widetilde{W}^{(M)}$ and we equate ranks with scores. That is, we let $M = L$ and we have one video of each type: $m_\ell \equiv 1$. Since videos and types are one and the same, we can index them by either m or ℓ; we choose to use m. The first column of $\widetilde{W}^{(M)}$ is given precisely by

$$\widetilde{W}_{m1}^{(M)} = \frac{w_m}{\sum_{m'=1}^{M} w_{m'}} \propto w_m \propto m^{-\alpha}, \tag{3.5}$$

where we assume the weights are given by the Zipf(α) distribution. Absorbing the constant of proportionality into γ_t (so we use $\widetilde{\gamma}_t$), and writing $p_{t,m}$ instead of $z_{t,\ell}$, we get the following expression for the probability a peer has video m at time t for the PL model with Zipf(α) weights with the direct recommendation rule:

$$p_{t,m} = \frac{1}{(\frac{1}{c}-1)e^{-\widetilde{\gamma}_t m^{-\alpha}}+1} \tag{3.6}$$

$$\sum_m p_{t,m} = Mc + t. \tag{3.7}$$

In spite of the approximations we have made, comparison with simulations in Section 3.6.3 suggests that (3.6)-(3.7) is rather accurate.

In the remainder of this section, we give a rigorous connection between the PL model and IC model in the large M limit. To make a ranking model look more like a score model it is natural to quantize ranks into a small number of scores as follows. Let $[L] = \{1, \ldots, L\}$ be the set of possible scores, with 1 denoting the best score and L the worst score, and let $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_L)$ be a probability vector with strictly positive entries. Let $\{I_1, \ldots, I_L\}$ be a partition of the interval $[0, 1]$ into intervals such that I_ℓ has length ρ_ℓ. Specifically, $I_1 = [0, \rho_1]$, $I_2 = (\rho_1, \rho_1 + \rho_2]$, $I_3 = (\rho_1 + \rho_2, \rho_1 + \rho_2 + \rho_3]$ and so on, and let $\psi : [0, 1] \to [L]$ be such that $\psi(r) = \ell$ for $r \in I_\ell$. Map a rank $r \in [M]$ into a score by applying ψ to the normalized rank, to get score $\psi(r/M)$. Let $G_\ell = \left\{m \in [M] : \frac{m}{M} \in I_\ell\right\}$, which is the set of ranks in $[M]$ that map to score ℓ, and let $m_\ell = |G_\ell|$. It is easy to check that $|m_\ell - \rho_\ell M| < 1$ for all ℓ. We assume the parameters (w_m) are such that w_m is decreasing in m, so that the items are indexed according to some master order. It is then natural to define the input score (or master score) of item m as $\psi(m/M)$. Thus, each item m has an input score $\psi(m/M)$ and a random output score $\psi(R(m)/M)$. The PL model and quantization function ψ then induce the crossover matrix $W^{(M)}$ obtained by tracking the scores of items before and after the PL channel: $W^{(M)}_{\ell\ell'} \triangleq \frac{1}{m_\ell}\sum_{m\in G_\ell} \mathbb{P}\left\{\psi(R(m)/M) = \ell'\right\}$. That is, $W^{(M)}_{\ell\ell'}$ is the probability that a randomly selected item with input score ℓ is given output score ℓ'. If $L << M$ then there is little dependence between the scores assigned to different items, given their input scores. The following theorem states a precise form of this observation for the case that the parameters w are given by a Zipf distribution: $w_m = (m/M)^{-\alpha}$ for some $\alpha > 0$. Since the PL model is invariant with respect to multiplicative scaling of the w's, this is equivalent to using parameters $w_m = m^{-\alpha}$.

Define $F_\infty(c) \triangleq \int_0^1 1 - \exp(-u^{-\alpha}c)du$ for $c \geq 0$, and

$$W_{\ell\ell'} \triangleq \frac{1}{\rho_\ell} \int_{I_\ell} \mathbb{P}\left\{F_\infty(x^\alpha Z) \in I_{\ell'}\right\} dx \tag{3.8}$$

for $\ell, \ell' \in [L]$, where Z has the exponential probability distribution with parameter one.[1] The following theorem is proved in Section 3.8.2. It says that for a fixed quantization function ψ, the quantized input and output scores for a fixed number of videos under the PL model converge jointly in distribution to the input and output scores for the same fixed number of videos for the IC model.

Theorem 3.5.1. *(Convergence of quantized* PL *to* IC*) Let $\alpha > 0$ and for $M \geq 1$ consider a random permutation R with the* PL *distribution with parameters M and $(w_i = (i/M)^{-\alpha} : i \in [M])$. Fix a quantizer function ψ with parameters L and $\boldsymbol{\rho}$ as described above. Fix $K \geq 1$ and let $A_1, \ldots, A_K$ denote K indices in $[M]$ selected uniformly at random without replacement. Note that $\psi(A_i/M)$ denotes the input score of item A_i and $\psi(R(A_i)/M)$ denotes the output score of item A_i.*

Let $(\widetilde{A}_1, \ldots, \widetilde{A}_K, \widetilde{B}_1, \ldots, \widetilde{B}_K)$ denote a vector of random variables such that the K pairs $(\widetilde{A}_k, \widetilde{B}_k)$, $1 \leq k \leq K$, are mutually independent, and for each $k \in [K] : \widetilde{A}_k$ has distribution $\boldsymbol{\rho}$ and $\mathbb{P}\left\{\widetilde{B}_k = j | \widetilde{A}_k = i\right\} = W_{ij}$ for $i, j \in [L]$.

Then $\left(\psi\left(\frac{A_1}{M}\right), \ldots, \psi\left(\frac{A_K}{M}\right), \psi\left(\frac{R(A_1)}{M}\right), \ldots, \psi\left(\frac{R(A_K)}{M}\right)\right)$ converges in distribution to $(\widetilde{A}_1, \ldots, \widetilde{A}_K, \widetilde{B}_1, \ldots, \widetilde{B}_K)$ as $M \to \infty$.

Theorem 3.5.1 suggests a way to analyze the performance of the PL model with a large number of peers and videos by approximating it by an IC model. The idea is to adopt a quantizer function ψ, compute the corresponding limiting crossover matrix W, and then use W in the mean field fluid limit analysis given for the IC model.

Our experience with this approach and comparison with simulations showed that the approximation improved as the number of levels L of the quantizer increases; see Section 3.6.2. That led to the proposal to use $M = L$ described at the beginning of the section.

[1] The formula given for W is equivalent to $W_{\ell\ell'} \triangleq \frac{1}{\rho_\ell} \int_{I_\ell \times I_{\ell'}} q(x,y)dxdy$ where, for each x fixed, $q(x,\cdot)$ is the pdf of the random variable $F_\infty(x^\alpha Z)$. The function q is in a sense the continuum limit of W as $L \to \infty$ with $\max_\ell \rho_\ell \to 0$.

3.6 Performance for Scoring Model

This section shows calculations and simulations that illustrate the mean field and fluid limit convergence, and also illustrate how performance of the PL model can be analyzed using the IC model. Then, the performance of the fluid approximation is compared with the upper bound of Section 3.2. The direct recommendation rule is assumed throughout. The model parameter values are the same as used in [8] and Section 2.5: 1000 peers, 1000 videos, 30 random videos seeded per peer, Zipf parameter $\alpha = 2.25$, with the exception that the buffer space here is assumed to be unlimited. The happiness of a peer at a given point in time is the fraction of videos it has with personal score L^* or better, among all videos with personal score L^* or better, where L^* is such that the expected number of videos with personal score L^* or better is 50. Each curve in each plot is the system happiness vs. time for one simulation run.

3.6.1 Direct Recommendation Rule with Binary Scores

To illustrate the mean field and fluid limit convergence for the IC model, we consider the case of $L = 2$ and $\boldsymbol{\rho} = (0.050, 0.950)$. The 2×2 crossover matrix W used is given by (3.8) for $\alpha = 2.25$ and $\boldsymbol{\rho}$. Namely,

$$W = \begin{pmatrix} 0.7419 & 0.2586 \\ 0.0136 & 0.9864 \end{pmatrix}.$$

Four performance curves are shown in Figure 3.1. One is the happiness for a simulation of the entire system of 1000 peers, 1000 videos, with 50 type 1 videos. The second curve is the fluid approximation determined by (3.3), (3.4), and W, as explained just after (3.4). The third curve is the performance predicted by a hybrid between the pure mean field model and the fluid limit of the mean field model, obtained as follows. Since $L = 2$, the more reduced state of a peer has the form $(z_1, z_2) \in \{0, \ldots, 50\} \times \{0, \ldots, 950\}$. For the hybrid calculation we track the distribution of z_1 which becomes a 51-state Markov chain and we represent z_2 by a deterministic real number updated by drift analysis (similar to (3.13)). Thus, for each time step, the tagged peer and contacted per have a random number of type 1 videos and a deterministic number of type 2 videos, and the probability the tagged peer downloads a type 1 video is calculated to get the peer distribution for the next time step. The fourth curve is the upper bound of Section 3.2.

Examination of Figure 3.1 shows close agreement between the simulation and the two numerically computed approximations to system happiness. In addition, the direct recommendation rule with binary scores performs nearly as well as the upper bound under IC model.

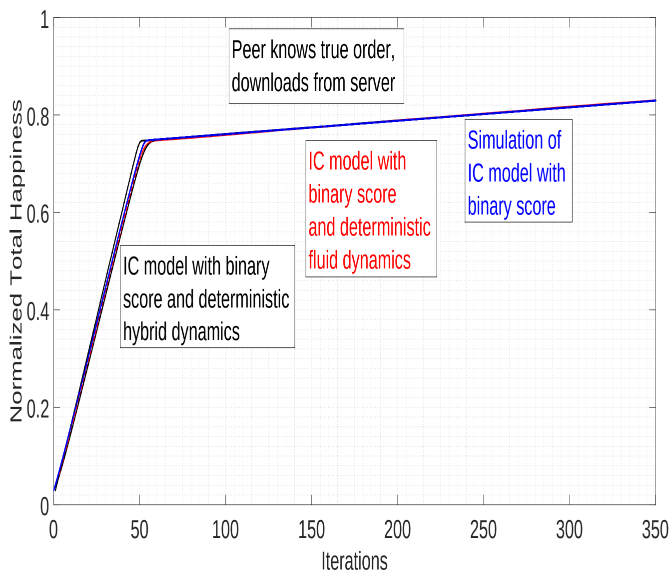

Figure 3.1: Video content collection performance with direct recommendation in a system of 1000 peers, 1000 videos, unlimited storage size, and binary scores under IC model

3.6.2 Direct Recommendation Rule with Many Scores

Figure 3.2 shows the performance curves for the direct recommendation rule under IC model for four different numbers of scores, $L = 2, 3, 4, 1000$. For $L = 2, 3, 4$ the crossover matrix W is given by (3.8) with $\alpha = 2.25$ and for $L = 1000$ the crossover matrix is given by (3.5). In addition, the upper bound of Section 3.2 is shown in the plot for $L = 1000$. The happiness is calculated using the fluid approximation determined by (3.3), (3.4), and W. The corresponding choices of $M\boldsymbol{\rho}$, giving the number of videos in each score group, are $(50, 950)$, $(50, 50, 900)$, $(25, 25, 50, 900)$, and $(1, 1, \ldots, 1)$, respectively.

It can be seen in the plot that the performance curves for small values of L are approximately piecewise linear, just as the curves in Figure 3.1 are. Moreover, the curve for $L = 1000$ essentially coincides with the simulated performance for the PL model with direct recommendation rule pictured in Figure 2.5 (not shown). In addition, the direct recommendation rule with $L = 1000$ performs nearly as well as the upper bound under IC model.

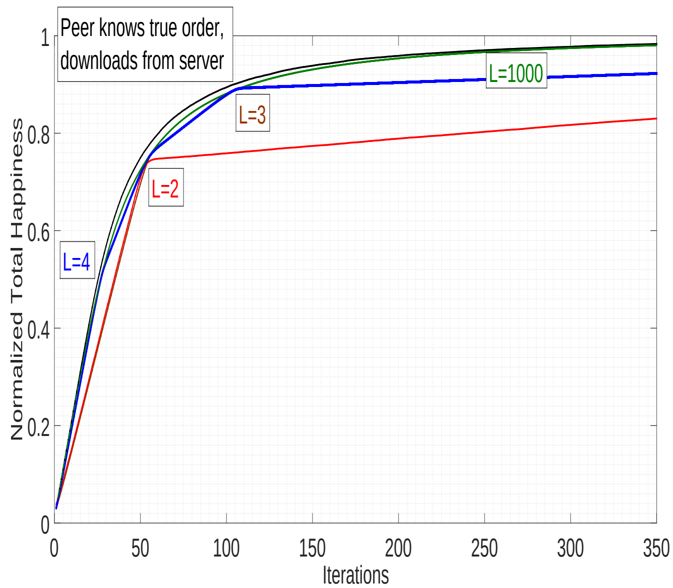

Figure 3.2: Video content collection performance with direct recommendation in a system of infinitely many peers, 1000 videos, unlimited storage size, and $L-ary$ scores for various L, under IC model, calculated by fluid limit

3.6.3 Recovering Performance of PL Model

As mentioned in the previous section, the fluid approximation for the IC model with $L = 1000$ gives an excellent prediction for the system happiness of the PL model. In fact, the fluid approximation predicts well the performance of the PL model in fine detail. To illustrate this, Figure 3.3 shows the fraction of peers that have video 1, 10, 20, 50, 100, or 250, respectively, as a function of time. The blue curves are from the simulation of the full PL system. The smoother red curves are calculated directly by (3.6)-(3.7).

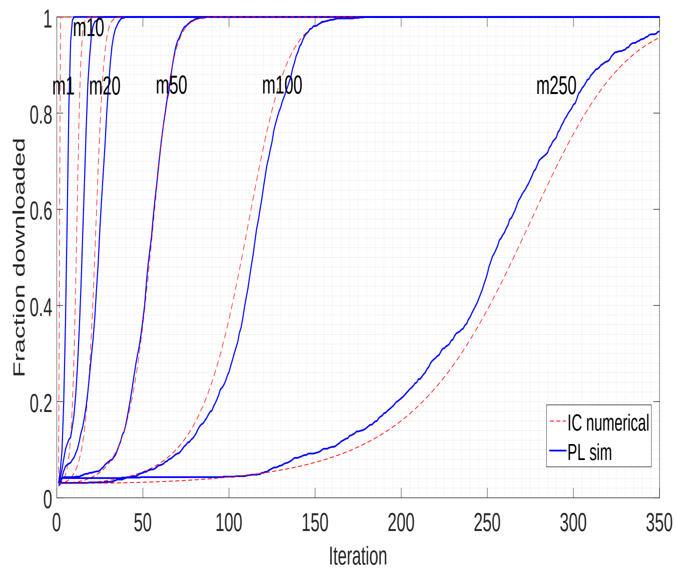

Figure 3.3: Fraction of videos downloaded with direct recommendation in a system of 1000 peers, 1000 videos, 100 storage size, and $k = 50$ under PL model (simulated) and IC model (numerically calculated)

3.7 Summary of Results

In the single-cluster regime with homogeneous population and IC scoring model, the direct recommendation rule is analyzed and extended to large system scaling. We applied Kurtz's theorem in the fluid limit where the number of peers and number of videos go to infinity. With several steps of simplification based on symmetry in the fluid limit and the LLN, we were able to perform an exact asymptotic analysis of the direct recommendation rule. We proved a theorem on a relationship between PL model and IC model under the fluid limit as the number of peers and number of videos go to infinity. Using the insights gained from the relationship between PL model and IC model, we provided a simple explicit approximation formula for the performance of the IC model under the direct recommendation rule. We found the calculated performance of the direct recommendation rule under the IC or PL models to very closely match simulations, and the performance nearly matches the upper bound.

3.8 Proofs of Limit Theorems

3.8.1 Proof of Theorem 3.4.1

To begin, we show that the distribution $\mu_0^{(M)}$ of the initial state Z_0 of the tagged peer concentrates on a small subset of the state space. A random variable X has the *hypergeometric distribution* with parameters n, k_1, k_2 if it has the following interpretation. If k_2 balls are drawn uniformly at random *without replacement* from an urn initially containing n balls, of which k_1 are red, then X denotes the number of red balls drawn. Note that $\mathbb{E}[X] = \frac{k_1 k_2}{n}$. Hoeffding [29] showed that this distribution is convex order dominated by the distribution that would result by sampling with replacement, namely, by the binomial distribution, Binom $\left(k_2, \frac{k_1}{n}\right)$. In particular, since the square function is convex, $\mathsf{var}(X) \leq k_2(\frac{k_1}{n})(1 - \frac{k_1}{n}) \leq \mathbb{E}[X]$.[2]

By assumption, a peer initially has Mc videos, selected uniformly at random from among all M videos, and for each $\ell \in [L]$, there are $m_\ell = \rho_l M$ type ℓ videos (i.e. videos with master score ℓ). Thus, the number of type ℓ videos the tagged peer initially has, denoted by $Z_{0,\ell}$, has the hyperexponential distribution with parameters $M, Mc, M\rho_\ell$. By Hoeffding's bound mentioned above, $\mathbb{E}[\|Z_0 - \boldsymbol{\rho} Mc\|^2] = \sum_\ell \mathsf{var}(Z_{0,\ell}) \leq Mc$. Therefore, by the Markov inequality, for any $\delta > 0$, $\mathbb{P}\{\|Z_0 - \boldsymbol{\rho} Mc\| \geq \delta M\} \leq \frac{c}{\delta^2 M} \stackrel{M\to\infty}{\longrightarrow} 0$. That is, $\frac{Z_0}{M} \to \boldsymbol{\rho} c$ in probability as $M \to \infty$. This concentration result implies that the initial state of any peer is greater than or equal to $\boldsymbol{\rho} Mc/2$ coordinatewise with probability converging to one. Since the states of peers are nondecreasing with time, it follows that

$$\inf_{t\geq 0} \left\{ \sum_{z \in S^{(M)}: z \geq \boldsymbol{\rho} Mc/2} \mu_{t,z}^{(M)} \right\} \stackrel{M\to\infty}{\longrightarrow} 1. \tag{3.9}$$

Note that $\sum_\ell \varphi_\ell(x, y) = 1$ so that $\sum_\ell b_{t,\ell} = c + t$ for $0 \leq t \leq 1 - c$, and $b_{1-c} = \boldsymbol{\rho}$. Therefore, to prove the theorem it suffices to show that for

[2] Hoeffding's result also implies that Chernoff bounds for binomial distributions also hold for the hypergeometric distribution. In particular, if X has a hypergeometric probability distribution, the Chernoff bounds for binomial X imply (see Theorems 4.4 & 4.5 in [30]): $\mathbb{P}\left\{\left|\frac{X}{\mathbb{E}[X]} - 1\right| \geq \epsilon\right\} \leq 2e^{-\epsilon^2 \mathbb{E}[X]/3}, \quad \forall 0 \leq \epsilon \leq 1$. For simplicity in this thesis, we will stick with bounds based on variance rather than exponential moments.

any fixed $\epsilon > 0$, $\sup_{0 \le t \le M(1-c-\epsilon)} \|\frac{Z_t}{M} - b_{t/M}\| \to 0$. To this end, fix $\epsilon > 0$ for the remainder of this proof. Consider the following subset of $S^{(M)}$: $\widetilde{S}^{(M)} = \left\{z \in \mathbb{Z}_+^L : m_\ell c/2 \le z_\ell \le m_\ell \text{ and } \sum_\ell z_\ell \le (1-\epsilon)M\right\}$. We only need to consider the system up until time $M(1-c-\epsilon)$ so that the upper constraint, $\sum_\ell z_\ell \le (1-\epsilon)M$, will not be violated by the state process of any peer. This, together with (3.9), implies:

$$\inf_{0 \le t \le M(1-c-\epsilon)} \left\{ \sum_{z \in \widetilde{S}^{(M)}} \mu_{t.z}^{(M)} \right\} \stackrel{M\to\infty}{\longrightarrow} 1. \tag{3.10}$$

In analogy with $S^{(M)}$ and $\widetilde{S}^{(M)}$, let:

$$\begin{aligned} S^{(\infty)} &= \{x \in \mathbb{R}_+^L : 0 \le x_\ell \le \rho_\ell\} \\ \widetilde{S}^{(\infty)} &= \{x \in \mathbb{R}_+^L : c\rho_\ell/2 \le x_\ell \le \rho_\ell \text{ and } \sum_\ell x_\ell \le 1-\epsilon\}. \end{aligned}$$

For $x, y \in \widetilde{S}^{(\infty)}$, the denominator of $\varphi(x,y)$ is greater than or equal to $\epsilon c \min_\ell \rho_\ell W_{\ell 1}/2$, and hence, bounded away from zero. It follows that φ and its first derivatives over $\widetilde{S}^{(\infty)} \times \widetilde{S}^{(\infty)}$ are bounded and continuous. Let $\widetilde{\varphi}$ be defined with domain $S^{(\infty)}$ such that $\widetilde{\varphi} = \varphi$ over the set $\widetilde{S}^{(\infty)} \times \widetilde{S}^{(\infty)}$, and $\widetilde{\varphi}$ and its first derivatives are bounded and continuous over $S^{(\infty)} \times S^{(\infty)}$. Since $b_t \in \widetilde{S}^{(\infty)}$ for $0 \le t \le 1-c-\epsilon$, we can also view b as the solution to $\dot{b} = \widetilde{\varphi}(b,b)$ with $b_0 = c\boldsymbol{\rho}$.

We show in this paragraph that

$$\sup_{z,z' \in \widetilde{S}^{(M)}} \left\| \Phi^{(M)}(z,z') - \varphi\left(\frac{z}{M}, \frac{z'}{M}\right) \right\| \to 0 \text{ as } M \to \infty. \tag{3.11}$$

Since $z, z' \in \widetilde{S}^{(M)}$ implies $\frac{z}{M}, \frac{z'}{M} \in \widetilde{S}^{(\infty)}$, the meaning of (3.11) is unchanged if φ is replaced by $\widetilde{\varphi}$. To verify the claim, fix $z, z' \in \widetilde{S}^{(M)}$ and consider the number of videos of some type ℓ eligible for transfer, assuming the tagged peer has state z and the contacted peer has state z'. The number of type ℓ videos that (i) the contacted peer has and (ii) the tagged peer does not have, denoted by N_ℓ, has the hyperexponential distribution with mean $\frac{(M\rho_\ell - z_\ell)z'_\ell}{M\rho_\ell}$. In turn, given N_ℓ, the number of type ℓ videos eligible for transfer has the $\text{Binom}(N_\ell, W_{\ell 1})$ distribution, with mean $\frac{(M\rho_\ell - z_\ell)z'_\ell W_{\ell 1}}{M\rho_\ell} \ge (M\rho_\ell - z_\ell)\frac{cW_{\ell 1}}{2}$, where we used the fact $z'_\ell \ge M\rho_\ell c/2$. The sum of these means is greater than or equal to $(M - \sum_\ell z_\ell)\frac{c\min_\ell W_{\ell,1}}{2} \geq \frac{M(1-\epsilon)c\min_\ell W_{\ell,1}}{2}$, where we used the fact $\sum_\ell z_\ell \leq (1-\epsilon)M$. Moreover, by the Chebychev inequalities for hyperexponential and binomial random variables, the numbers of eligible videos of each type are within δM of their means with high probability, for arbitrarily small $\delta > 0$. Therefore, the distribution of the type of the video to be transferred is, in the limit $m \to \infty$, proportional to the means, namely, it is given by $\varphi\left(\frac{z}{M}, \frac{z'}{M}\right)$. Moreover, the estimates involved are uniform in $z, z' \in \widetilde{S}^{(M)}$, implying (3.11) as claimed.

In analogy to the definition of $\Phi^{(M,1,t)}$, let

$$\varphi^{(M,1,t)}(x) \triangleq \sum_{z' \in \widetilde{S}^{(M)}} \widetilde{\varphi}\left(x, \frac{z'}{M}\right) \mu^{(M)}_{t,z'}. \tag{3.12}$$

Let $(B_t^{(M)} : 0 \leq t \leq M(1-\epsilon-t))$ denote the deterministic, discrete-time trajectory in $\mathbb{R}_+^L$ defined recursively as follows:[3]

$$B_t^{(M)} = \sum_{s=0}^{t-1} \varphi^{(M,1,s)}\left(\frac{B_s^{(M)}}{M}\right) + Mc\boldsymbol{\rho}. \tag{3.13}$$

Next, we use arguments from the classical theory of limits of Markov processes, to show that the trajectory of the Markov process Z closely follows B with high probability. Adding and subtracting various terms and arranging them yields:

$$\begin{aligned} Z_t = &\sum_{s=0}^{t-1} \varphi^{(M,1,s)}\left(\frac{Z_s}{M}\right) \\ &+ \sum_{s=0}^{t-1} \left[\Phi^{(M,1,s)}(Z_s) - \varphi^{(M,1,s)}\left(\frac{Z_s}{M}\right)\right] + Mc\boldsymbol{\rho} + \mathcal{M}_t. \end{aligned} \tag{3.14}$$

$\mathcal{M}_t = \sum_{s=0}^{t-1}\left(Z_{s+1} - Z_s - \Phi^{(M,1,s)}(Z_s)\right) + \mathcal{M}_0$, and $\mathcal{M}_0 = Z_0 - Mc\boldsymbol{\rho}$. The process $(\mathcal{M}_t)$ is a mean zero Martingale. As explained near the beginning of the proof, Hoeffding's result yields $\mathbb{E}\left[\|\mathcal{M}_0\|^2\right] \leq Mc\sum_\ell \rho_\ell(1-\rho_\ell) \leq Mc$. Also, $\|\mathcal{M}_{t+1} - \mathcal{M}_t\|^2 = \|Z_{t+1} - Z_t - \Phi^{(M,1,t)}(Z_t)\|^2 \leq 2$ with probability one. Furthermore, the increments of square integrable Martingales are orthogonal

[3] By convention, $\sum_{s=0}^{t-1}$ is zero for $t = 0$.

random variables. Thus,

$$\mathbb{E}\left[\|\mathcal{M}_t\|^2\right] = \mathbb{E}\left[\|\mathcal{M}_0\|^2\right] + \sum_{s=0}^{t-1} \mathbb{E}\left[\|\mathcal{M}_{s+1} - \mathcal{M}_s\|^2\right] \leq Mc + 2t.$$

Therefore, using Doob's L^2 inequality,

$$\mathbb{E}\left[\sup_{0\leq s\leq M(1-\epsilon-c)} \|\mathcal{M}_s\|^2\right] \leq 4\mathbb{E}\left[\|\mathcal{M}_{M(1-\epsilon-c)}\|^2\right] \leq 8M.$$

Hence, if $\delta > 0$ is a small fixed constant, by the Markov inequality,

$$\mathbb{P}\left\{\sup_{0\leq s\leq M(1-\epsilon-c)} \|\mathcal{M}_s\| \geq \frac{\delta M}{2}\right\} \leq \frac{32}{\delta^2 M} \to 0 \text{ as } M \to \infty.$$

By (3.10), (3.11), and the boundedness of Φ and $\widetilde{\varphi}$, it follows that the norm of the quantity in square brackets in (3.14) is at most $\delta/2$ for M sufficiently large. Hence, (3.14) yields that, with probability converging to one as $M \to \infty$,

$$Z_t = \sum_{s=0}^{t-1} \varphi^{(M,1,s)}\left(\frac{Z_s}{M}\right) + Mc\boldsymbol{\rho} + e_t, \tag{3.15}$$

where $\|e_t\| \leq \delta M$ for $0 \leq t \leq M(1-\epsilon-c)$. Since $\widetilde{\varphi}$ has bounded derivatives, the derivatives of $\varphi^{(M,1,s)}$ are uniformly bounded over all s, so that $\varphi^{(M,1,s)}$ is c_L-Lipschitz continuous for all s for some finite constant c_L. So, subtracting the respective of sides of (3.13) from (3.15) and using $\|e_t\| \leq \delta M$ yields that

$$\|Z_t - B_t^{(M)}\| \leq \frac{c_L}{M} \sum_{s=0}^{t-1} \|Z_s - B_s^{(M)}\| + \delta M.$$

Thus, by induction on t, $\|Z_t - B_t^{(M)}\| \leq (1+\frac{c_L}{M})^t \delta M \leq \exp(\frac{c_L t}{M})\delta M$. Therefore, with probability converging to one as $M \to \infty$,

$$\sup_{0\leq t\leq M(1-\epsilon-c)} \left\|\frac{Z_t}{M} - \frac{B_t^{(M)}}{M}\right\| \leq e^{c_L}\delta. \tag{3.16}$$

Next, we revisit the definition (3.12) of $\varphi^{(M,1,t)}$. It shows that $\varphi^{(M,1,t)}(x)$ is obtained by averaging out y in $\widetilde{\varphi}(x, y)$ using the probability distribution of

Z_t/M. But (3.16) shows that Z_t/M is close to $B_t^{(M)}/M$ with high probability. It follows that for any $\delta > 0$, $\left\| \varphi^{(M,1,t)}(x) - \varphi\left(x, \frac{B_t^{(M)}}{M}\right) \right\| \to 0$ as $M \to \infty$, uniformly over $x \in \widetilde{S}^{(\infty)}$ and $t \in [0, M(1-\epsilon-c)]$. Applying this observation to (3.13) then yields

$$\frac{B_t^{(M)}}{M} = \frac{1}{M} \sum_{s=0}^{t-1} \varphi\left(\frac{B_s^{(M)}}{M}, \frac{B_s^{(M)}}{M}\right) + c\boldsymbol{\rho} + e'_t, \tag{3.17}$$

where $\max_{0 \le t \le M(1-\epsilon-c)} \|e'_t\| \to 0$ as $M \to \infty$. Introduce time scaling by letting $\tau = t/n$ and $b_\tau^{(M)} = \frac{B_{M\tau}^{(M)}}{M}$ if τ is a multiple of $1/M$ and defining $b_\tau^{(M)}$ elsewhere on $[0, 1-\epsilon-c]$ by linear interpolation. Since φ is bounded the functions $b_\tau^{(M)}$ are uniformly Lipschitz continuous, and hence by the Arzelà-Ascoli theorem, any subsequence has a convergent sub-subsequence. By (3.17), any limit trajectory $b^{(\infty)}$ satisfies

$$b_\tau^{(\infty)} = \int_0^\tau \varphi(b_\sigma^{(\infty)}, b_\sigma^{(\infty)}) d\sigma + c\boldsymbol{\rho}.$$

This integral equation has a unique solution, namely, $b^{(\infty)} = b$, so the entire sequence $b^{(M)}$ converges uniformly to b as $M \to \infty$. So, $\sup_{0 \le t \le M(1-\epsilon-c)} \| \frac{B_t^{(M)}}{M} - b_{t/M}\| \to 0$, which, together with (3.16), implies Theorem 3.4.1.

3.8.2 Proof of Theorem 3.5.1

Using the exponential representation for the PL distribution with parameter vector $w_1, \ldots, w_M$, we can assume that $R(m)$ is the rank of X_m among the independent random variables $X_1, \ldots, X_M$, such that X_m is exponentially distributed with rate parameter w_m for each m. Let $\widehat{F}_M(c) \triangleq \frac{1}{M} \sum_{m \in [M]} I_{\{X_m \le c\}}$. That is, $\widehat{F}_M$ is the empirical cumulative distribution function of $(X_m : m \in [M])$. Notice that for any item m, $\frac{R(m)}{M} = \widehat{F}_M(X_m)$. That is, the rank of m, normalized by division by M, is gotten by applying the function $\widehat{F}_M$ to X_m. So $\widehat{F}_M$ is a *stochastic ranking scale* that maps X values into normalized ranks. The following lemma shows that for large M, $\widehat{F}_M$ is well approximated by F_∞, which acts as a deterministic ranking scale.

Lemma 3.8.1. *Let* $w_m = (m/M)^{-\alpha}$ *and*

$F_\infty(c) = \int_0^1 1 - \exp(-u^{-\alpha}c)du$. *Then for any* $\delta > 0$,[4]

$$\mathbb{P}\left\{\sup_{c\geq 0} |\widehat{F}_M(c) - F_\infty(c)| \leq \delta\right\} \to 1. \tag{3.18}$$

Proof. By its definition, $\widehat{F}_M(c)$ for a fixed value of c, is the average of M independent random variables, where the m^{th} random variable has the Bernoulli distribution with parameter $1 - \exp(-(m/M)^{-\alpha}c)$. Since Riemann sums converge to integrals and can be bounded by integrals:

$$\mathbb{E}\left[\widehat{F}_M(c)\right] = \frac{1}{M}\sum_{m=1}^{M}(1 - \exp(-(m/M)^{-\alpha}c) \to F_\infty(c)$$
$$\mathsf{var}(\widehat{F}_M(c)) \leq \frac{1}{M^2}\sum_{m=1}^{M}(1 - \exp(-(m/M)^{-\alpha}c) \leq \frac{F_\infty(c)}{M} \leq \frac{1}{M},$$

it follows that, by the Chebychev inequality, $\widehat{F}_M(c) \to F_\infty(c)$ in distribution as $M \to \infty$ for any fixed c. Select c_i so that $F_\infty(c_i) = \delta i/2$ for integers i with $1 \leq i < 2/\delta$. Since the number of values of i is fixed, it follows that $\mathbb{P}\{E_\delta\} \to 1$ where $E_\delta \triangleq \{\max_i |\widehat{F}_M(c_i) - F_\infty(c_i)| \leq \delta/2\}$. Furthermore, since $\widehat{F}_M$ is a nondecreasing function and F_∞ is a continuous, strictly increasing function, both with range $[0,1]$, the event E_δ implies that $\sup_{c\geq 0} |\widehat{F}_M(c) - F_\infty(c)| \leq \delta$, so that (3.18) holds as claimed. □

Proof of Theorem 3.5.1. We shall use a particular representation of $(\widetilde{A}_k, \widetilde{B}_k)$. Let $U_1, \ldots, U_K$ each be uniformly distributed on the interval $[0,1]$ and let $Z_1, \ldots, Z_K$ each be exponentially distributed with mean one, and suppose $U_1, \ldots, U_K, Z_1, \ldots, Z_K$ are mutually independent. Then we can assume that $\widetilde{A}_k = \psi(U_k)$ and $\widetilde{B}_k = \psi \circ F_\infty(U_k^\alpha Z_k)$. That is, with this representation, the pairs $(\widetilde{A}_k, \widetilde{B}_k)$, $1 \leq k \leq K$ are independent, the variables $\psi(U_k)$ have distribution $(\rho_1, \ldots, \rho_K)$, and $\mathbb{P}\left\{\widetilde{B}_k = \ell' | \widetilde{A}_k = \ell\right\} = W_{\ell\ell'}$ (to prove the last property, use the fact that given $\widetilde{A}_k = i$, U_k is uniformly distributed over I_i).

A version of the continuous mapping theorem of measure theory (e.g. see [31]) states that if a sequence of random vectors V_n converges in distribution to a random vector V_∞, and if ϕ is a Borel measurable function such that $\mathbb{P}\{V_\infty \in D_\phi\} = 0$, where D_ϕ is the set of discontinuity points of ϕ, then

[4]This is a variation of the Glivenko-Cantelli theorem; here the X's are not identically distributed.

$\phi(V_n)$ converges in distribution to $\phi(V_\infty)$. In view of the continuous mapping theorem, it suffices to prove

$$\left(\frac{A_1}{M}, \dots, \frac{A_K}{M}, \frac{R(A_1)}{M}, \dots, \frac{R(A_K)}{M}\right)$$
$$\xrightarrow{d} (U_1, \dots, U_K, F_\infty(U_1^\alpha Z_1), \dots, F_\infty(U_K^\alpha Z_K)) \tag{3.19}$$

as $M \to \infty$. Indeed, applying the function $\phi : [0,1]^{2K} \to [L]^{2K}$ defined by $\phi(x_1, \dots, x_{2K}) = (\psi(x_1), \dots, \psi(x_{2K}))$ to the vectors on each side of (3.19) makes the distributions of the corresponding sides match the distributions of the vectors in the last sentence of Theorem 3.5.1. Since the random vector on the righthand side of (3.19) has a joint probability density function, the probability the vector is in D_ϕ is zero.

It remains to prove (3.19), and for that we use a coupling argument. First, the random variable A_k for $k \in [K]$ has the same distribution as $\lceil U_k M \rceil$. So we can let $A_k = \lceil U_k M \rceil$ for all k in the event that the random variables $\lceil U_k M \rceil$ are distinct, which has probability of converging to one as $M \to \infty$. That is, we can assume that the random vector $(A_1, \cdots, A_K)$ for each M and the random vector $(U_1, \dots, U_K)$ are all constructed on the same probability space so that

$$\mathbb{P}\left\{A_k = \lceil U_k M \rceil : 1 \le k \le K\right\} \ge \left(1 - \frac{K}{M}\right)^K \overset{M\to\infty}{\to} 1.$$

In particular, if follows that $\frac{A_k}{M} \to U_k$ in the sense of convergence in probability, for each k. For the PL model, once the items $A_1, \dots, A_K$ are selected, exponential random variables (i.e. $X_{A_k} : 1 \le k \le K$) must be generated for these items, as well as for all the other items. We can assume without loss of generality that these exponential random variables are taken to be $\left(\frac{A_k}{M}\right)^\alpha Z_k : 1 \le k \le K$, where $(Z_1, \dots, Z_K)$ are the same exponential random variables used for the representations of the $\widetilde{B}$'s and appearing in (3.19). Then $\frac{R(A_k)}{M} = \widehat{F}_M\left(\left(\frac{A_k}{M}\right)^\alpha Z_k\right)$. Recall that $\mathbb{P}\{E_\delta\} \to 0$ as $M \to \infty$, and

$|\widehat{F}_M(c) - F_\infty(c)| \leq \delta$ for all c on the event E_δ. Thus, on the event E_δ,

$$\begin{aligned}
&\left|\frac{R(A_k)}{M} - F_\infty(U_k^\alpha Z_k)\right| \\
&= \left|\widehat{F}_M\left(\left(\frac{\lceil U_k M\rceil}{M}\right)^\alpha Z_k\right) - F_\infty(U_k^\alpha Z_k)\right| \\
&\leq \delta + \left|F_\infty\left(\left(\frac{\lceil U_k M\rceil}{M}\right)^\alpha Z_k\right) - F_\infty(U_k^\alpha Z_k)\right|. \qquad (3.20)
\end{aligned}$$

By continuity of F_∞, the second term in (3.20) converges to zero almost surely as $M \to \infty$, and hence also in the sense of convergence in probability. And since $\delta > 0$ is arbitrary, it follows that $\left|\frac{R(A_k)}{M} - F_\infty(U_k^\alpha Z_k)\right| \to 0$ in probability for each k. Thus (3.19) holds in the sense of convergence in probability, and hence, also in distribution. $\square$

CHAPTER 4

THE MULTI-CLUSTER FRAMEWORK

Real world populations are usually heterogeneous. In order to model heterogeneous preferences, we consider a multi-cluster model for correlated score assignments by peers and use insights gained from the single-cluster model in Chapter 2 and Chapter 3 to propose distributed recommendation rules in the multi-cluster regime.

We briefly compare centralized and distributed recommendation for multi-cluster systems. For a centralized system, where all peers' partial scoring preferences are accessible, the central tracker can apply clustering algorithms like the K-means heuristic to solve for the clustering problem, which includes estimating the cluster centers' score vectors. Once each peer knows its cluster center score vector, it downloads videos from the central server according to the preference order of its cluster center scores because of the stochastic dominance property of PL with Zipf distribution, stated in Theorem 2.2.1.

For a distributed system which is the focus of this work, there are problems of both limited preference information and limited video availability. Each peer can be either selfish or helpful. If a peer is selfish, it first collects preference information and estimates its cluster center score vector. Then it downloads videos from its contacted peers according to the preference order of its estimated cluster center scores. If a peer is helpful, it gathers more preference information from contacted peers and passes on preference information to contacted peers. Then it performs clustering on the collected preferences to identify every cluster center score vector. It downloads videos based on the estimated cluster center scores to allow videos which are commonly preferred among multiple clusters to disseminate faster; i.e., peers cooperate to emulate a central tracker.

4.1 A Generative Cluster Model for Heterogeneous Peers

We consider a simple closed heterogeneous system in a mixture model. The number of peers, N, and the number of videos, M, are fixed over time. The number of clusters, K, and the partition of peers also remain fixed over time. The peers are assumed to be indexed by $[N] \triangleq \{1, \ldots, N\}$ and the videos are assumed to be indexed by $[M]$. Let n_k be the number of peers in the kth cluster.

Suppose there are L possible scores, $[L] = \{1, ..., L\}$, where each video belongs to one of the master score set, $G_1, \ldots, G_L$. Given a $L \times L$ stochastic matrix W^α, for any video in G_ℓ, a cluster center score vector assigns score ℓ' to the video with probability $W^\alpha_{\ell\ell'}$. The scores assigned by all cluster centers to all videos are assumed to be independent, given the types of the videos. Each cluster k for $k \in [K]$ has an intrinsic cluster scores of videos, $i_k : [M] \to [L]$. Similarly, given the cluster center's score for each video, the personal scores of peers are generated independently using another crossover probability matrix, W^β, giving rise to a two-stage $\alpha - \beta$ independent crossover ($\alpha\beta\mathsf{IC}$) model. We adopt the convention that a lower numerical score indicates a more preferred video.

If A, B are finite multisets of $\mathbb{R}$, $A \succ B$ (A is better than B) indicates that $|A| \geq |B|$ and $a_{[i]} \leq b_{[i]}$ for $1 \leq i \leq |B|$, where $a_{[1]} \leq \ldots \leq a_{[|A|]}$ denotes the ordered elements of A and $b_{[1]} \leq \ldots \leq b_{[|G|]}$ denotes the ordered elements of B. The happiness function of a peer at time t is defined as $H_n(t) = f(\{i_n(m)|m \in S_n(t)\})$, where $f : 2^{[M|} \to \mathbb{R}$ is assumed to be nondecreasing in the $\succ$ order, and $2^{[M]}$ denotes the set of subsets of $[M]$.

4.2 Upper Bound on System Performance

To obtain an upper bound on the happiness of a peer, consider an idealized system in which the peer has access to a server that can provide any video, and for which a genie reveals extra information. If the genie revealed to the peer its cluster center score vector, then the peer's optimal recommendation rule is to download the videos in the order of increasing cluster center scores. This follows from the first theorem in the previous analysis. Based on our mixture model assumptions, this upper bound performance of the multi-cluster model is statistically the same as the upper bound performance of the single-cluster model in Section 3.2.

4.3 Clustering: Similarity Measures and Error

This section explores similarity measures between two partial preference vectors and investigates whether two such vectors can be inferred to be from peers in the same cluster based on their similarity.

4.3.1 The Potential for Clustering Assuming the $\alpha\beta$IC Model

In multi-cluster systems, the rate at which a peer contacts another peer from the same cluster is reduced in proportion to the fraction of peers in the contacting peer's cluster. To provide better recommendations when contacted peers are in different clusters, it might be beneficial if peers use recommendation rules more complex than direct recommendations, specifically, rules that perform clustering.

With the assumption that clusters are planted, the clustering problem is the task of identifying which cluster each peer belongs to. If PL with Zipf distribution induced crossover probability matrices are used, we can roughly identify the difficulty of the clustering problem given the Zipf parameters. For simplicity, we denote the Zipf parameters for the crossover probability matrices W^{α} and W^{β} as α and β respectively. When α is large, cluster centers are more correlated than when α is small. If β is small, then personal preferences of the same cluster are more scattered than when β is large. In this combination of α and β, the clustering problem seems to be more difficult than other combinations of α and β. In contrast, when α is small and β is large, the clustering problem seems to be easier than other combinations of α and β.

Note that if α is sufficiently large, although it is difficult to cluster preferences, it is easy to find a near optimal recommendation rule because the multi-cluster recommendation problem approximately reduces to the single-cluster recommendation problem that we have previously studied. If β is sufficiently small, it is also difficult to cluster preferences, because the preferences are very scattered. In this case, the lack of correlations among preferences cannot provide enough useful inference information, so recommendation rules might not work well. Since some choices of α and β present difficulty in clustering, we choose to analyze the distributed recommendation system under $\alpha < \beta$ with moderate values.

4.3.2 Generic Similarity Measures for Clustering

Before analyzing the recommendation rules for the distributed system, we study clustering measures on the mixture model to gain insight into clustering. One way to make clustering decisions is to use distance or similarity measures and a threshold. The calculations of distance or similarity measures described here do not depend on W^{α}, W^{β}, or clusters sizes (fraction of peers in each cluster), although the thresholds will depend on these quantities. We define the following distance and similarity measures which are commonly used for scores (references), the normalized Euclidean distance and the cosine similarity. The normalized Euclidean distance between two partial preference vectors i_u and i_v is defined as

$$d(i_u, i_v) = \sqrt{\frac{\sum_{m \in S_u \cap S_v} (i_u(m) - i_v(m))^2}{|S_u \cap S_v|}}$$

and the cosine similarity between two partial preference vectors is defined as

$$s(i_u, i_v) = \frac{\sum_{m \in S_u \cap S_v} i_u(m) i_v(m)}{\sqrt{\sum_{m \in S_u \cap S_v} i_u(m)^2} \sqrt{\sum_{m \in S_u \cap S_v} i_v(m)^2}},$$

where S_u and S_v are the nonzero indices of the two partial preference vectors. Because of the heavy tailed Zipf distribution and the choices of α and β, in particular $\alpha < \beta$, preferences under different clusters are less correlated and thus more likely to yield larger differences in video scores. Normalized Euclidean distance, which is more sensitive to the magnitude of the vectors, would tend to result in a larger separation between two preferences from different clusters compared to cosine similarity, which is more sensitive to the angle between the vectors. For example, suppose in one case peer u has $(1, 5)$ and peer v has $(5, 1)$ as scores for two common videos and in another case peer u has $(2, 10)$ and peer v has $(10, 2)$ as scores for the two videos. In both cases the cosine similarities are the same but the Euclidean distances are different. According to our model assumptions and choices of α and β, the second pair of preferences, $(2, 10)$ and $(10, 2)$, are less likely to belong to the same cluster compared to the first pair, $(1, 5)$ and $(5, 1)$. An example of Euclidean distance and cosine similarity is plotted in Figure 4.1 and Figure 4.2 between the partial preference vector of a peer from cluster 1 and the partial preference vectors of 1000 peers from clusters 1 to 10, ordered with intervals of 100 peers in each cluster. Notice that the decision line on the cosine similarity plot and the decision line on the normalized Euclidean distance plot intersect nearly identical sets of peers, so only the distance metric will be used in the following sections.

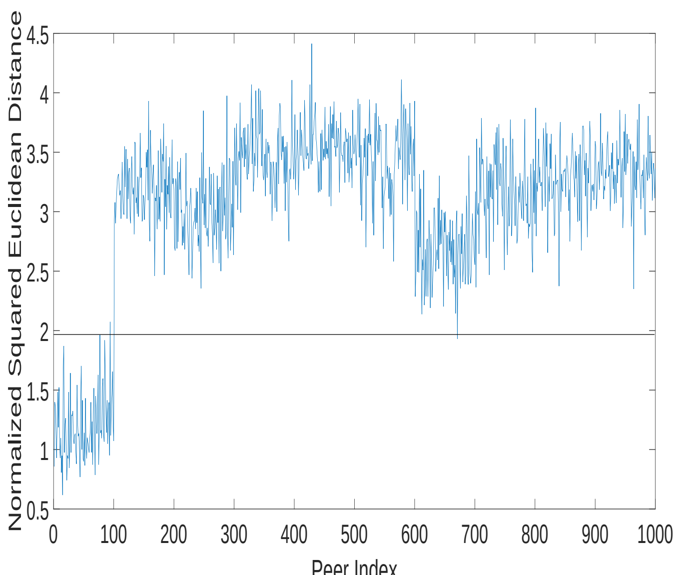

Figure 4.1: Normalized squared Euclidean distance between a peer from cluster 1 and the rest

4.3.3 Model-Based Similarity Measure for Clustering

Suppose W^α, W^β, prior distribution of master scores (fraction of videos in each master score) and prior distribution of clusters (fraction of peers in each cluster) are known, we can make clustering decisions by hypothesis testing. The multi-cluster mixture model is shown in Figure 4.3. The α channel between the master score vector and each cluster center score vector is characterized by crossover probability matrix W^α and the β channel

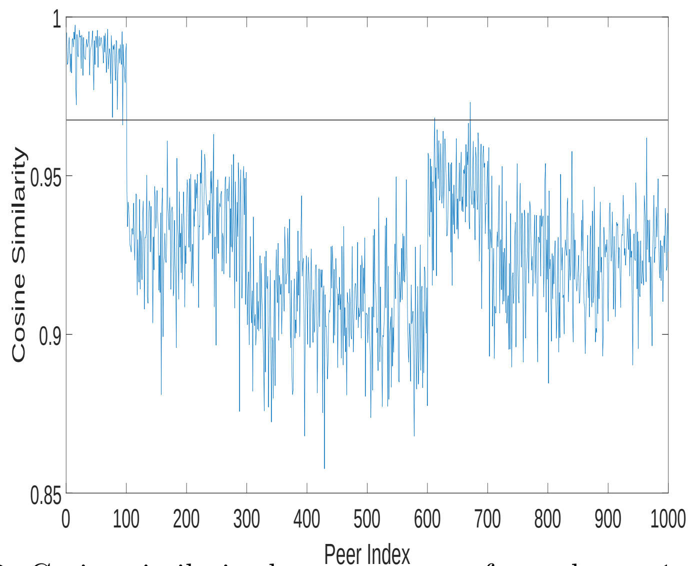

Figure 4.2: Cosine similarity between a peer from cluster 1 and the rest

between a cluster center score vector and each personal preference vector is characterized by W^{β}. We let H_1 denote the hypothesis that the contacted peer v is in the same cluster as the contacting peer u and H_0 denote the hypothesis that the contacted peer v is in a different cluster.

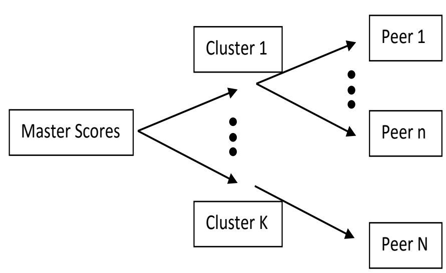

Figure 4.3: Multi-cluster channel

When the contacting peer and the contacted peer share common videos, we can calculate the posterior probability of the two hypotheses. Because of the independent crossover channels, the following is true under either hypothesis: The observed scores of the contacted peer are conditionally independent given the scores of the contacting peer. The conditional probabilities of the observed scores of the contacted peer given the scores of the contacting peer under each hypothesis is thus the product of the conditional probability for each video over all the common videos. Looking at a single video, the conditional probability of observing score j from the contacted peer when the contacting peer rates it score i under hypotheses H_1 and H_0 are shown as follows:

$$H_1 : P_{H_1}(j|i) \quad = \quad \sum_{\ell} \frac{W^{\beta}_{\ell i}\rho(\ell)W^{\beta}_{\ell j}}{\sum_{\ell'}\rho(\ell')W^{\beta}_{\ell' i}} \tag{4.1}$$

$$\begin{aligned} &H_0 : P_{H_0}(j|i) \\ &= \sum_m \left(\sum_o \left(\sum_{\ell} \frac{W^{\beta}_{\ell i}\rho(\ell)}{\sum_{\ell'}\rho(\ell')W^{\beta}_{\ell' i}} \frac{W^{\alpha}_{o\ell}\rho(o)}{\sum_{o'}\rho(o')W^{\alpha}_{o'\ell}} \right) W^{\alpha}_{om} \right) W^{\beta}_{mj}, \end{aligned} \tag{4.2}$$

where $\rho(\ell)$ is the prior distribution of score ℓ for $1 \leq \ell \leq L$. Recall that PL induced matrices preserve the distributions of scores, so the distribution of personal scores, cluster center scores and master scores are the same.

The posterior probabilities of H_1 or H_0 can be calculated as the product of the conditional probability in (4.1) or (4.2) and the prior distribution of clusters. When the two peers share multiple videos, the posterior probability of each hypothesis is multiplicative over the set of common videos, shown as follows:

$$H_1 : P(H_1|\vec{i},\vec{j}) = \frac{P_{H_1}(\vec{j}|\vec{i})P(H_1)}{\rho(\vec{j})} = \frac{\prod_m P_{H_1}(j(m)|i(m)) \cdot P(H_1)}{\rho(\vec{j})} \tag{4.3}$$

$$H_0 : P(H_0|\vec{i},\vec{j}) = \frac{P_{H_0}(\vec{j}|\vec{i})P(H_0)}{\rho(\vec{j})} = \frac{\prod_m P_{H_0}(j(m)|i(m)) \cdot P(H_0)}{\rho(\vec{j})}. \tag{4.4}$$

4.3.4 Clustering Error

In multi-cluster systems, preference information is more useful if the preferences belong to peers who are similar to the contacting peer. Under a flash start regime, peers share few common videos with each other. Since clustering is based on the personal scores of the common videos shared between the peers, it takes a long time for them to collect enough preference information for the contacting peer to cluster. If the peers could view a common set of test videos, it would speed up the clustering phase, which would allow peers to quickly identify preferences that belong to their own clusters.

Having multiple common videos between peers decreases the decision error. In hypothesis testing, the decision error based on a single common video can be upper bounded in terms of the Bhattacharyya coefficient [32]. Given that the contacting peer rates the video score i, the Bhattacharyya coefficient is $\rho_i \triangleq \int \sqrt{P_{H_0}(\ell|i)P_{H_1}(\ell|i)}d\ell$, where $P_{H_h}(\ell|i)$ is the conditional probability that the contacted peer rates the video score l under hypothesis $h \in \{0, 1\}$ described in (4.1) and (4.2). The upper bound for the decision error is given as $P_e(\pi|i) \leq \sqrt{\pi_0\pi_1}\rho_i$, where π_h is the prior distribution of hypotheses H_0 and H_1. Based on the choice of W^α and W^β, the decision error based on a single common video can be calculated with respect to the personal scores of the contacting peer. For example, when $\alpha = 0.8$ and $\beta = 2.25$, the Bhattacharyya coefficient is

$$\rho_i = [0.6706, 0.7982, 0.8464, 0.8782, 0.9108, 0.9379, 0.9623, 0.9763, 0.9783, 0.9554],$$

corresponding to the contacting peer's personal scores $i = [1, 2, ..., 10]$ respectively.

To upper bound the decision error on multiple common videos, notice that only the Bhattacharyya coefficient needs to be changed to $P_{H_h}(\vec{\ell}|\vec{i})$ since the observed personal scores of the videos from the contacted peer are conditionally independent given the hypothesis and the personal scores of the contacting peer. The Bhattacharyya coefficient for a set of common videos is thus the product of the ρ_i's, $P_{H_h}(\vec{\ell}|\vec{i}) = \prod_m P_{H_h}(\ell(m)|i(m))$. Because the Bhattacharyya coefficient is multiplicative, the upper bound of the decision error decreases exponentially. Peers can make clustering decisions with high accuracy after viewing just a small set of common videos. In the following section, a small set of common videos are given to each peer to view initially.

4.4 Recommendation Rules Based on Stored Partial Preference Vectors

One performance benchmark is when peers do not cluster and follow the simple direct recommendation rule as if the contacted peer is in the same cluster. This recommendation rule performs poorly for the multi-cluster system, because the rate of downloading videos from other peers with similar preferences is very low. This section describes recommendation rules for which peers collect partial preference vectors of other peers.

Each peer stores a set of partial preference vectors that is updated each time the peer contacts another peer. We consider two possibilities for how aggressively peers collect partial preference vectors:

Linear Collection of Partial Preference Vectors

When a peer contacts another peer, the contacted peer's preference vector is first added to the set of vectors stored by the contacting peer. If the number of stored partial preference vectors exceeds a maximum storage size, the vector least similar to the contacting peer's preference vector is removed. We call this the least similar preference (LSP) storage management policy. Since at most one partial preference vector is added per contact, the number of stored vectors increases linearly until the maximum storage size is reached.

Exponential Collection of Partial Preference Vectors

With just a small set of partial preference vectors, few videos have sufficiently many scores for accurate score prediction. To increase the accuracy of recommendations, more preference information can be collected. For this method, when a peer contacts another peer, the entire set of stored partial preference vectors from the contacted peer is collected. This allows the amount of preference information at each peer to grow exponentially in time until the maximum storage size is reached. Each peer essentially emulates the server's role in collecting preference information and estimating its personal preference, except the peer is still restricted to collect videos from the peers it contacts. Because there are more available partial preference vectors as a result of the exponential collection of partial preference vectors, it makes sense to increase the storage capacity at each peer while still using the LSP storage management policy. Also, more restrictive thresholds can be applied in recommendation rules to yield more accurate score prediction. Recommendation rules based on the lists of stored partial preference vectors are described next.

4.4.1 Nearest Stored Preference Recommendation Rule

A simple rule using the list of stored partial preference vectors is the nearest stored preference recommendation rule. For this rule, peer clustering and video selection are done in a two-step procedure, followed each time a peer contacts another peer.

Step 1) Based on the fact that preference information from another peer is more valuable if it is more correlated with the contacting peer's preference, the contacting peer is likely to learn more about its own preferences from more correlated preferences. In the first step, the strength of correlation between the contacting peer's preference vector and each of its stored partial preference vectors is calculated by the distance metric or the model-based similarity measure, described in Section 4.3.

Step 2) The contacting peer selects the most similar partial preference vector from its stored set. The videos with numerically higher (bad) scores are filtered out from this partial preference vector, so only videos with sufficiently good scores are considered. Then, the contacting peer downloads from the contacted peer's available videos following the order of the filtered partial preference vector. If no videos can be selected due to the unavailability at the contacted peer, the next most similar partial preference vector is selected and filtered. This process continues until the contacting peer finds a video to download from the contacted peer.

4.4.2 Bayesian Recommendation Rules

Bayesian Recommendation Rule with Soft Clustering

We next describe a two-step procedure in which all sufficiently similar preference vectors from the contacting peer's stored set are identified and then combined to estimate the cluster center scores for the contacting peer. The first step is calculating the strength of correlation between the contacting peer's preference vector and each of its stored partial preference vectors. The second step is combining stored scoring preference information and estimating which video to download. Details are as follows:

Focusing on a contacting peer, denoted by peer u, let PS be peer u's set of stored partial preference vectors. We split each partial preference vector from the set PS into two subvectors as follows: one with preferences for videos that peer u has rated and the other with preferences for videos that peer u has not rated. Let PSR denote the set of subvectors for videos peer u has rated and let $PSNR$ denote the set of subvectors for videos peer u has not rated. Let i_u be the partial preference vector of peer u and c_u be peer u's cluster center score vector. Following the same partition as above, let $i_{u,r}$ and $c_{u,r}$ denote the parts of i_u and c_u restricted to the domain of videos peer u has rated and let $i_{u,nr}$ and $c_{u,nr}$ denote the parts of i_u and c_u restricted to the domain of videos peer u has not rated yet.

Step 1) For each preference vector $i_s \in PS$, the probability i_s is in the same cluster as peer u is calculated. Recall that for a given i_s, $H_{0,s}$ is the hypothesis that i_s is in a different cluster from peer u, and $H_{1,s}$ is the hypothesis that i_s is in the same cluster as peer u. The strength of correlation between peer u's preference vector and each stored partial preference vector i_s is calculated independently using the posterior probabilities of hypotheses $H_{0,s}$ and $H_{1,s}$. Note that to calculate the posterior probabilities of hypotheses $H_{0,s}$ and $H_{1,s}$ for each stored partial preference vector, it suffices to use the corresponding partial preference vector from PSR and $i_{u,r}$. Thus, we apply (4.3) and (4.4) and substitute $\vec{i} = i_{u,r}$ and $\vec{j} = i_{s,r} \in PSR$ to get $P(H_{1,s}|i_{s,r}, i_{u,r})$ and $P(H_{0,s}|i_{s,r}, i_{u,r})$.

Step 2) Peer u downloads video with the highest posterior expected happiness. The posterior expected happiness for a particular video m not yet viewed by peer u can be calculated from the posterior distribution of peer u's personal score of the video and the happiness function. The posterior distribution of peer u's personal score of each video is based on the posterior distribution of peer u's cluster center score of the video and the crossover probability matrix W^{β}. Therefore, given the happiness function and W^{β}, in order to determine which video to download, it suffices to calculate the posterior distribution of peer u's cluster center score for each video. We explain how to do that next.

Following (4.1) and (4.2) with a slight modification, given the cluster center score of video m is a, the likelihood of observing score b from a stored partial preference vector i_s under hypothesis $H_{1,s}$ or $H_{0,s}$ respectively, is shown as follows:

$$H_{1,s} : P_{H_{1,s}}(b|a) = W^{\beta}_{ab} \tag{4.5}$$

$$H_{0,s} : P_{H_{0,s}}(b|a) = \sum_{a'} \left(\sum_{o} \frac{W^{\alpha}_{oa}\rho(o)W^{\alpha}_{oa'}}{\sum_{o'} \rho(o')W^{\alpha}_{o'a}} \right) W^{\beta}_{a'b}. \tag{4.6}$$

Combining (4.5) and (4.6) for the video, given peer u's cluster center score of video m is a, the likelihood of observing score b for video m in vector i_s is as follows:

$$\begin{aligned} P(b|a, i_{s,r}, i_{u,r}) &= P_{H_{0,s}}(b|a) \cdot P(H_{0,s}|i_{s,r}, i_{u,r}) \\ &\quad + P_{H_{1,s}}(b|a) \cdot P(H_{1,s}|i_{s,r}, i_{u,r}) \end{aligned} \tag{4.7}$$

Combining the entire set of stored partial preference vectors for video m, given peer u's cluster center score of video m is a, the likelihood of observing the scores over the set of stored partial preference vectors is as follows:

$$P(PSNR(m)|a, PSR, i_{u,r}) = \prod_{i_s \in PS} P(i_{s,nr}(m)|a, i_{s,r}, i_{u,r}).$$

Applying the Bayes rule, the posterior probability of score $c_{u,nr}(m)$ of peer u's cluster center score for an unrated video m, given the observations from the set of stored partial preference vectors, is as follows:

$$P(c_{u,nr(m)} = a|PS, i_{u,r}) = \frac{P(PSNR(m)|a, PSR, i_{u,r}) \cdot \rho(a)}{\prod_{i_s \in PS} \rho(i_{s,nr}(m))}. \tag{4.8}$$

Combining (4.8) with the happiness function and W^{β}, we obtain the expected happiness for each video that the contacting peer has not rated. The peer then downloads an available video from the contacted peer with the highest expected happiness. Here we have made the approximations that each peer's stored partial preference vectors are independent and scores of videos are independent when calculating the posterior probability distribution of the cluster center scores.

Bayesian Recommendation Rule with Hard Clustering

The previous Bayesian recommendation rule can be modified to become a Bayesian recommendation rule with hard decision. The first step of the two-step procedure is the same as before; the strength of correlation between the contacting peer's preference vector and each of its stored partial preference vectors is calculated independently. The second step is similar to the nearest stored preference recommendation rule because this rule also filters information. This recommendation rule makes a hard decision and ignores the stored partial preference vectors that are not sufficiently similar to the contacting peer, i.e. the stored partial preference vectors that are not likely to be in the same cluster as the contacting peer. The second step is also similar to the Bayesian recommendation rule with soft decision because this rule combines stored partial preference vectors to estimate which video to download, but only on the set of stored partial preference vectors that are similar to the contacting peer. Details are as follows:

Step 1) For each preference vector $i_s \in PS$, the probability i_s is in the same cluster as peer u is calculated using the posterior probability of hypotheses $H_{0,s}$ and $H_{1,s}$, i.e. $P(H_{0,s}|i_{s,r}, i_{u,r})$ and $P(H_{1,s}|i_{s,r}, i_{u,r})$.

Step 2) Declare i_s is from the same cluster as peer u if $P(H_{1,s}|i_{s,r}, i_{u,r})$ exceeds a threshold. The threshold is set by the Neyman-Pearson criteria, being sufficiently large to restrict the probability of false alarm to be low, so the remaining set of stored partial preference vectors is very likely to be in the same cluster as the contacting peer. Let $PSS \subseteq PS$ denote the remaining set of stored partial preference vectors.

Peer u downloads video with the highest posterior expected happiness. The posterior expected happiness for a particular video m not yet viewed by peer u can be calculated from the posterior distribution of peer u's cluster center score, the happiness function, and W^{β}. In contrast to the soft clustering rule, this recommendation rule calculates the posterior distribution of peer u's cluster center score for each video based on PSS, which is assumed to contain partial preference vectors only in peer u's cluster after the hard decision. Then for $i_s \in PSS$, we use $P(H_{0,s}|i_{s,r}, i_{u,r}) = 0$ and $P(H_{1,s}|i_{s,r}, i_{u,r}) = 1$ and (4.7) becomes $P(b|a, i_{s,r}, i_{u,r}) = P_{H_{1,s}}(b|a)$. Combining the set PSS for video m, given peer u's cluster center score of video m is a, the likelihood of observing the scores over PSS is as follows:

$$P(PSSNR(m)|a) \;\; = \prod_{i_s \in PSS} P_{H_{1,s}}(i_{s,nr}(m)).$$

Applying Bayes rule, the posterior probability of score $c_{u,nr}(m)$ of peer u's cluster center score for an unrated video m, given the observations from the set of stored partial preference vectors in the same cluster as the contacting peer, is as follows:

$$P(c_{u,nr(m)} = a|PSS, i_{u,r}) \;\; = \;\; \frac{P(PSSNR(m)|a) \cdot \rho(a)}{\prod_{i_s \in PSS} \rho(i_{s,nr}(m))}. \qquad (4.9)$$

Combining (4.9) with the happiness function and W^{β}, we obtain the expected happiness for each video that the contacting peer has not rated. The peer then downloads an available video from the contacted peer with the highest expected happiness. Here we have made the approximations that each peer's stored partial preference vectors are independent and scores of videos are independent when calculating the posterior probability distribution of the cluster center scores.

4.5 Multi-Cluster Aware Global List Recommendation Rule

In all previously discussed recommendation rules, each peer collects multiple partial preference vectors. The stored partial preference vectors are then processed to determine which video to download. A minimalistic approach without the need to store partial preference vectors is to recursively combine them. We call this recommendation rule the multi-cluster aware global list recommendation rule, and we denote the aggregate of the partial preference vectors the global list, which follows from Cruz [8]. Each peer maintains one global list, denoted by g_u. A peer's global list is a set of L vectors of tallies, in which the mth tally of the ℓth vector, $g_u(\ell, m)$, stores the number of partial preference vectors observed so far that belong to the same cluster as the contacting peer and have rated score l for video m.

The multi-cluster aware global list recommendation rule is a two-step procedure executed when one peer contacts another. The first step is for the contacting peer to update its global list. The second step is for the contacting peer to estimate which video to download. Focusing on a contacting peer u, details are as follows:

Step 1) The strength of correlation between peer u's preference vector and the contacted peer's preference vector is calculated by the distance metric or the model-based similarity measure. A hard decision is made using a threshold to ignore the contacted peer's preference vector if it belongs to a different cluster. If the contacted peer's preference vector belongs to the same cluster as peer u, peer u's global list is updated as follows: for each video $m \in [M]$, $g_u(\ell, m)$ is incremented by one if the contacted peer rates video m score $\ell \in [L]$.

Step 2) Similar to before, peer u downloads the video with the highest posterior expected happiness. The posterior expected happiness of a particular video m not yet viewed by peer u can be calculated from the posterior distribution of peer u's cluster center score, the happiness function, and W^{β}. The difference is that this recommendation rule calculates the posterior distribution of peer u's cluster center score for each video based on the L corresponding tallies from its global list, denoted by $g_u(m) = \{g_u(\ell, m) : \ell \in [L]\}$. Given peer u's cluster center score of video m is a, the likelihood of observing the L corresponding tallies is as follows:

$$P(g_u(m)|a) \quad = \quad \prod_{\ell \in [L]} P(\ell|a)^{g_u(\ell,m)}.$$

Applying the Bayes rule, the posterior probability distribution of score $c_{u,nr}(m)$ of peer u's cluster center score for an unrated video m, given the L corresponding tallies from its global list, is as follows:

$$P(c_{u,nr}(m) = a | g_u(m)) \quad = \quad \frac{P(g_u(m)|a) \cdot \rho(a)}{\prod_{\ell \in [L]} \rho(\ell)^{g_u(\ell,m)}}. \tag{4.10}$$

Combining (4.10) with the happiness function and W^{β}, we obtain the expected happiness for each video that the contacting peer has not rated. The peer then downloads an available video from the contacted peer with the highest expected happiness. Here we have made the approximation that the contacted peers are distinct so the contacting peer's global list is recursively updated without checking for duplicate partial preference vectors and scores of videos are independent when calculating the posterior probability distribution of the cluster center scores.

Note that the multi-cluster aware global list recommendation rule is almost exactly the same as the Bayesian recommendation rule with hard clustering, because both recommendation rules have the same hard clustering in step 1 and the same Bayesian recommendation rule to calculate the posterior distribution of each peer's cluster center scores. The subtle difference is in the Bayesian recommendation rule with hard clustering; each peer calculates the posterior distribution of its cluster center scores based on a set of stored partial preference vectors with LSP storage management policy. In the multi-cluster aware global list recommendation rule, because of the aggregated tallies, each peer effectively calculates the posterior distribution of its cluster center scores based on the partial preference vectors from all of its contacted peers that are sufficiently similar. The multi-cluster aware global list recommendation rule is the same as the Bayesian recommendation rule with hard clustering with unlimited storage of partial preference vectors and without LSP storage management policy.

4.6 Performance for Multi-Cluster Scoring Model

To compare the recommendation rules described in Section 4.4 along with the upper bound of Section 4.2 for application to the scoring model, we simulated them for the system parameters similar to the ones used in [8] and Section 3.6: 1000 peers, 1000 videos, and 30 random videos seeded per peer. In addition, 10 random common videos are seeded for each peer. Each peer obtains the preference vector of its contacted peer and stores at most 30 preference vectors with LSP storage management policy. For the generative cluster model, the Zipf parameters are $\alpha = 0$ and $\beta = 2.25$, so the cluster centers are well separated. We assign scores to the videos from $\{1, ..., L\}$ with $L = 10$ and distribution $\rho = (0.025, 0.025, 0.025, 0.025, 0.5, 0.5, 0.1, 0.1, 0.3, 0.3)$. The 10×10 crossover matrices W^α and W^β used are given by (3.8). The happiness of a peer at a given point in time is the fraction of videos it has with personal score $L^* = 2$ or better, so the expected number of videos with personal score $L^* = 2$ or better is 50. Each curve in each plot is the system happiness vs. time for one simulation run.

4.6.1 Benchmarks

Before simulating the proposed recommendation rules, we would like to check the impact of multi-clusters on the distributed recommendation system. To illustrate the difference between multi-cluster and single-cluster, two benchmarks are shown. Given a genie telling each peer its cluster center score vector, the first benchmark is the upper bound described in Section 4.2, when peers download according to their cluster center's score vectors from the server. This is the same upper bound shown in the single-cluster section. The second benchmark is when peers download according to the simple direct recommendation rule described in Section 4.4, which is the same rule described in the single-cluster section, but applied on the multi-cluster system. The performance curves are shown in Figure 4.4. We observe a large gap between the performance of the direct recommendation rule and the upper bound.

4.6.2 Nearest Stored Preference Recommendation Rule

Recall that the nearest stored preference recommendation rule stores preference vectors from all of the peer's previously contacted peers up to the storage constraint and performs a simple filtering algorithm. It is very similar to the direct recommendation rule because both recommendation rules utilize a simple greedy filtering algorithm. The major difference is that the nearest stored preference recommendation rule stores multiple preference vectors so

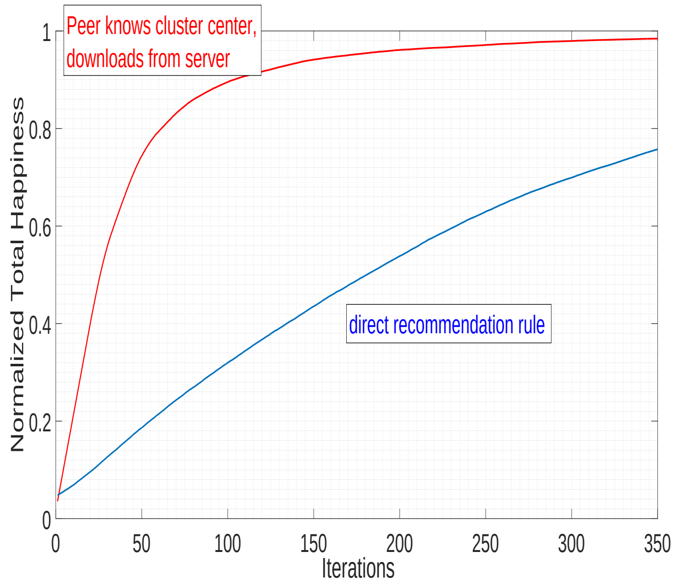

Figure 4.4: Video content collection performance with simple direct recommendation rule and upper bound in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, and $\alpha = 0$ and $\beta = 2.25$

videos can be selected based on a larger set of preference information. The comparison between the performance of these two simple recommendation rules under the single cluster regime ($\alpha = \infty$) is shown in Figure 4.5. We observe that the nearest stored preference recommendation rule achieves near optimal performance under the single cluster regime and achieves slightly better performance than the direct recommendation rule.

Figure 4.6 shows the happiness vs. time for the nearest stored preference rule based on linear collection of partial preference vectors. It appears there is a noticeable gap between its performance and the upper bound under the multi-cluster regime.

4.6.3 Bayesian Recommendation Rules

Figure 4.6 also shows the happiness vs. time for the Bayesian recommendation rule with soft and hard clustering based on linear collection of partial preference vectors. Recall that the two recommendation rules store preference vectors from all of the peer's previously contacted peers up to the storage constraint and apply complex filtering algorithms. These two recommendation rules are nearly identical and appear to have performance

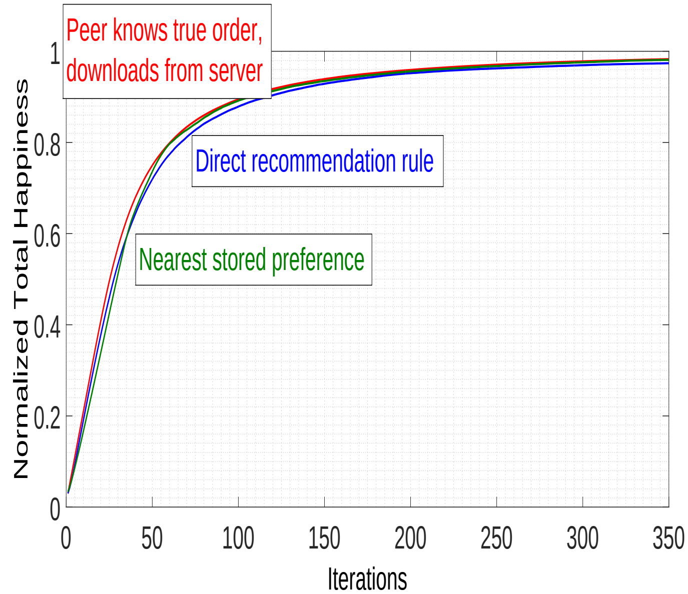

Figure 4.5: Video content collection performance with nearest stored preference recommendation rule in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, and $\alpha = \infty$ and $\beta = 2.25$

similar to that of the nearest stored preference rule.

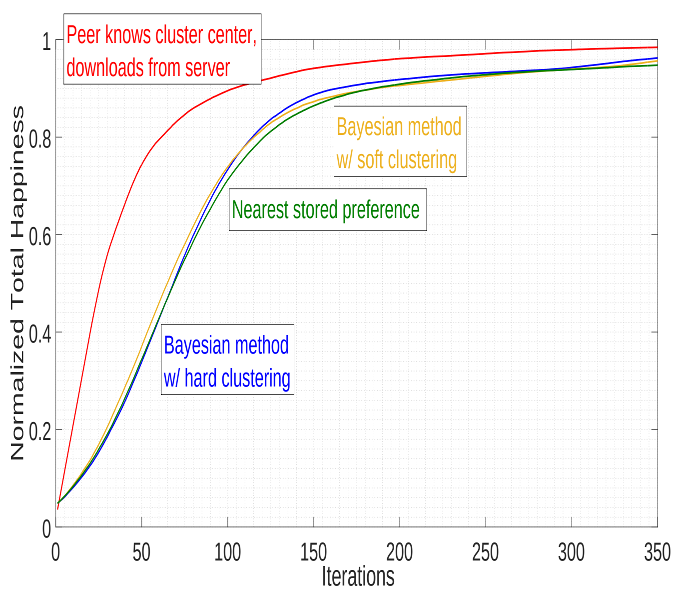

Figure 4.6: Video content collection performance with nearest stored preference recommendation rule and Bayesian recommendation rule with soft and hard clustering in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, and $\alpha = 0$ and $\beta = 2.25$

4.6.4 Multi-Aware Global List Recommendation Rule

Figure 4.7 shows the happiness vs. time for the multi-cluster aware global list recommendation rule based on linear collection of partial preference vectors. Recall that the recommendation rule requires only the space for one preference vector to store the aggregate preference information from the peer's previously contacted peers and applies complex filtering algorithm. It appears to have performance similar to that of the nearest stored preference rule.

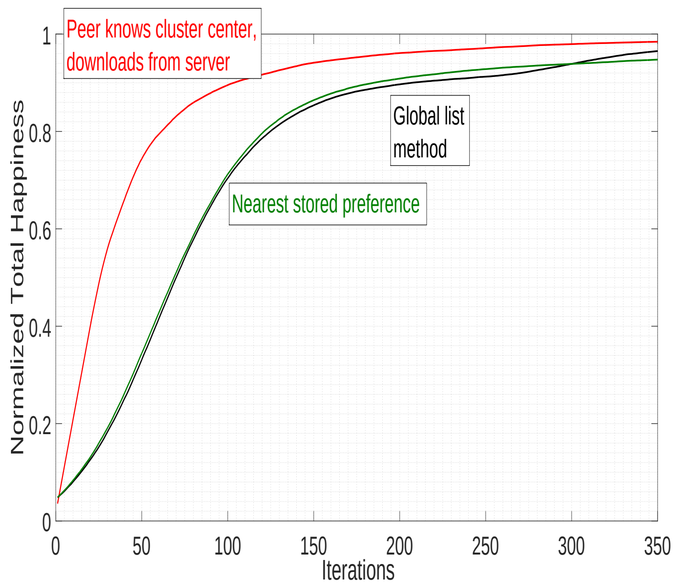

Figure 4.7: Video content collection performance with nearest stored preference recommendation rule and multi-cluster aware global list recommendation rule in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, and $\alpha = 0$ and $\beta = 2.25$

4.6.5 Cluster Correlations

To check the performance of the recommendation rules under more similar personal preference distribution under the generative cluster model, the Zipf parameter α is changed from 0 to 0.8 to yield correlated cluster centers. When α is 0, the cluster centers are uniformly random. When α is 0.8, the cluster centers are slightly correlated. Figure 4.8 shows the performance of the recommendation rules described in Section 4.4 when $\alpha = 0.8$ and based on linear collection of partial preference vectors. In the simulation, the nearest stored preference rule performs the best, displaying its robustness against varying personal preference distributions similar to the robustness of the direct recommendation rule in the single-cluster regime. The Bayesian recommendation rule performs slightly better with soft clustering than with the hard clustering. The global list recommendation rule performs the worst. The Bayesian recommendation rules and the global list recommendation rule did not perform well, perhaps because when cluster centers are closer together, preference information outside of the contacting peer's cluster is more likely to introduce noise in combining preference information to estimate which video to download. In contrast, for the nearest stored preference rule, preference information is not combined.

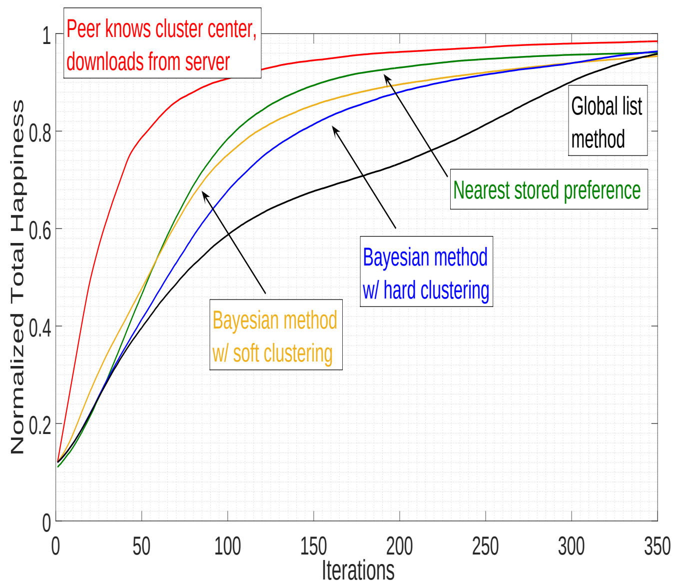

Figure 4.8: Video content collection performance with multi-cluster aware recommendation rules in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, and $\alpha = 0.8$ and $\beta = 2.25$

4.6.6 Exponential Collection of Partial Preference Vectors

To reduce the performance gap between the recommendation rules and the upper bound, partial preference vectors are collected exponentially in time. We relax the storage constraint for preference vectors and assume the bandwidth incurred from the exchange of preference information is negligible. Then, each peer collects both its contacted peers' personal preference vector and its contacted peers' stored partial preference vectors. Figure 4.9 shows the happiness vs. time for the multi-cluster aware recommendation rules with exponential collection of partial preference vectors described in Section 4.4. The three performance curves with exponential collection of partial preference vectors appear to be significantly closer to the upper bound compared to without exponential collection of partial preference vectors. Of the three recommendation rules, the Bayesian recommendation rule with soft clustering and that with hard clustering have similar performance and perform better than the nearest stored preference rule.

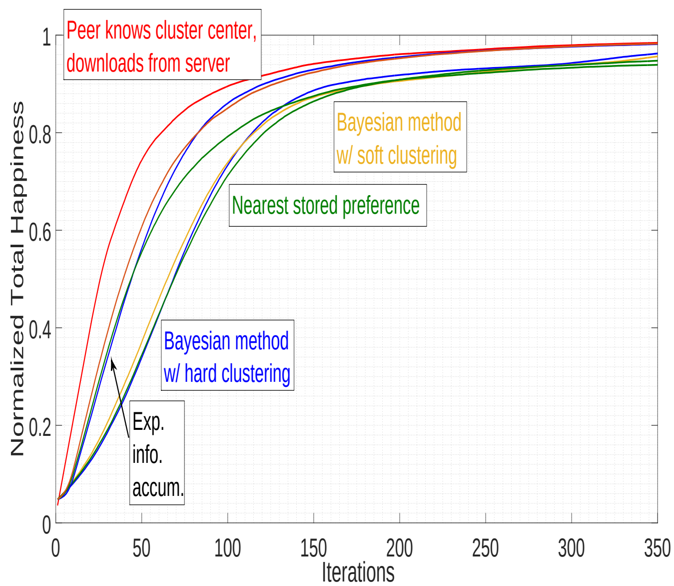

Figure 4.9: Video content collection performance with multi-cluster aware recommendation rules with exponential collection of partial preference vectors in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, and $\alpha = 0$ and $\beta = 2.25$

4.6.7 Graphical Structures

So far, all simulations are based on a fully connected network graph with uniformly random connections. Here, we explore graphical structures, where each peer's contacts are limited to its neighbors. To illustrate the effect of neighborhood structure, we consider the following two types of neighborhoods: a friendly neighborhood structure and a random neighborhood structure. In a friendly neighborhood structure, most of a peer's neighbors are from the same cluster. In a random neighborhood structure, each peer's neighbors belong to a random fixed subset of peers in the system.

For the friendly neighborhood structure, each peer has approximately 15 neighbors from its own cluster selected uniformly at random and 3 neighbors from other clusters selected uniformly at random. The performances of the nearest stored preference rule under the friendly neighborhood structure and the fully connected network graph with uniformly random connections are shown in Figure 4.10. Note that other recommendation rules have similar performance as the nearest stored preference rule, so they are omitted in the plot. It can be seen that the friendly neighborhood structure yields much better performance than fully connected network graph in the beginning, because preferred videos are more likely to be available from peers in the same cluster.

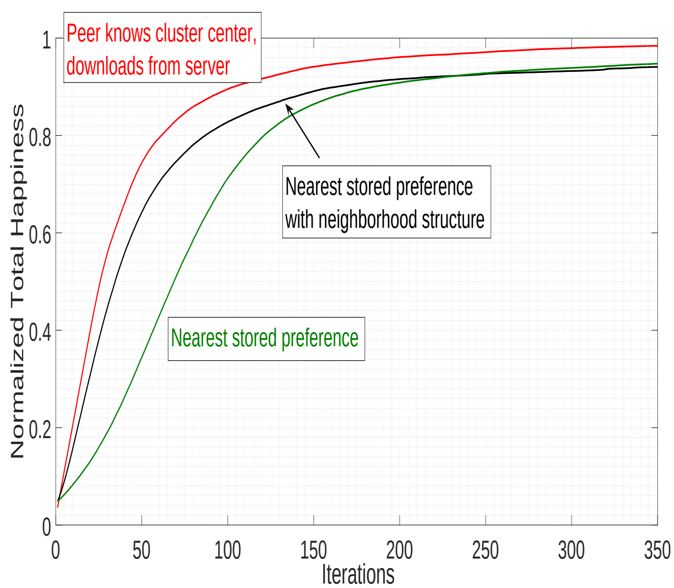

Figure 4.10: Video content collection performance with genie in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, $\alpha = 0$, $\beta = 2.25$, and a friendly neighborhood structure

For the structure with random neighborhoods, each peer has approximately 18 neighbors selected uniformly at random from all the peers. The performances of the nearest stored preference rule under the random neighborhood structure and the fully connected network graph with uniformly random connections are shown Figure 4.11. Note that other recommendation rules have similar performance as the nearest stored preference rule, so they are omitted in the plot. It can be seen that the random neighborhood structure yields much poorer performance than fully connected network graph, because preferred videos are less likely to be available.

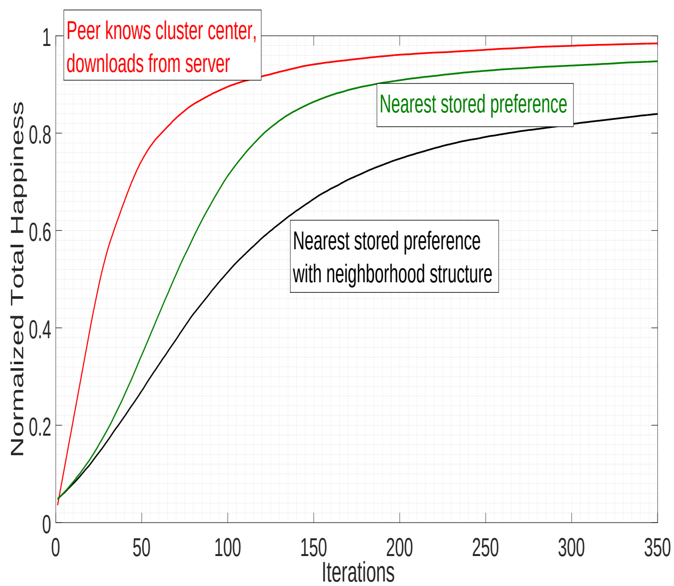

Figure 4.11: Video content collection performance with genie in a system of 1000 peers, 1000 videos, unlimited storage size, $L = 10$, $\alpha = 0$, $\beta = 2.25$, and a random neighborhood structure

4.7 Summary of Results

In the multi-cluster regime with heterogeneous population and IC scoring model, we analyzed the distributed recommendation system that jointly performs content collection and rank aggregation. We identified a performance upper bound carried over from the single-cluster regime. From the upper bound and a benchmark, we determined the necessity for clustering under the multi-cluster regime. We identified an approximate difficulty of clustering under various Zipf parameters. We then described similarity measures used for clustering, e.g. normalized Euclidean distance, cosine similarity, and model-based similarity using hypothesis testing. Based on the model-based similarity, we identified an upper bound on clustering error by calculating the Bhattacharyya coefficient. Using the intuition obtained from the upper bound on the clustering error, a small set of common videos are initially given to each peer in order to increase the clustering accuracy.

Four multi-cluster aware recommendation rules were proposed. With some modifications to the MLE rule, we proposed two deluxe recommendation rules that estimate the cluster center scores: the Bayesian recommendation rule with soft clustering and that with hard clustering. We proposed a simple greedy recommendation rule inspired by the direct recommendation rule, called the nearest stored preference rule. Lastly, we proposed a multi-cluster aware global list recommendation rule that is modified from Cruz's recommendation rule. In terms of accumulation of information, peers under the nearest stored preference rule and the Bayesian recommendation rules need to store partial preference vectors, while the multi-cluster aware global list recommendation rule does not. In terms of processing information, the nearest stored preference rule has the simplest filtering algorithm.

A main conclusion is that distributed content collection and rank aggregation is feasible. Under well separated cluster centers, the proposed recommendation rules all perform similarly, and the nearest stored preference rule is the simplest. Under more correlated cluster centers, the nearest stored preference rule works remarkably well, behaving similarly to the direct recommendation rule in the single-cluster regime with homogeneous population. In general, either exponential accumulation of partial preference vectors or neighbor assignments such that neighbors have similar preferences is sufficient for the nearest stored preference rule and Bayesian recommendation rules to achieve near optimal performance.

CHAPTER 5

CONCLUSION

This thesis aims to understand a distributed system with joint content collection and rank aggregation. For the single-cluster PL ranking model, we presented a trivial performance upper bound and a tighter performance upper bound using stochastic comparison. Using the intuition obtained from the performance upper bound, we applied Hunter's MM algorithm and proposed an elaborate recommendation rule using this algorithm. We reevaluated Cruz's recommendation rule and proposed more efficient variations of the rule that recursively combine partial rankings. Specifically, these are the linear weighting functions. We also proposed a simple greedy recommendation rule called direct recommendation and found in simulations that it is near optimal. The direct recommendation rule is also robust in the sense that it performs remarkably well over a broad range of system parameters.

For the single-cluster IC score model, we applied Kurtz's theorem in the fluid limit where the number of peers and number of videos go to infinity. With several steps of simplification based on symmetry in the fluid limit and the LLN, we were able to perform an exact asymptotic analysis of the direct recommendation rule. We also stated a relationship between PL model and IC model under the fluid limit. Using insights gained from this relationship, we provided a simple explicit approximation formula for the performance of the IC model under the direct recommendation rule. We also found in simulations that the numerical performance result is very good approximation for the direct recommendation rule under PL model in the fluid limit. In addition, the numerical result of the direct recommendation rule appears to be near optimal.

For the multi-cluster IC score model, we proposed four multi-cluster aware recommendation rules based on the insights gained from the study of recommendation rules under the single-cluster regime. We distinguished two aspects of a recommendation rule: accumulation of information and processing of information. We extended the MLE recommendation rule from Hunter's MM algorithm to two recommendation rules: the Bayesian recommendation rule with soft clustering and that with hard clustering. The Bayesian recommendation rules apply a complex filtering algorithm. We proposed a simple greedy recommendation rule similar to the direct recommendation rule, called the nearest stored preference rule, and found in simulations that its performance is relatively good. The nearest stored preference rule applies a simple filtering algorithm. Peers utilizing the nearest stored preference rule or the Bayesian recommendation rules store multiple preference vectors up to a storage constraint. We also extended the global list recommendation rule by Cruz to a recommendation rule called the multi-cluster aware global list recommendation rule. The multi-cluster aware global list recommendation rule applies a complex filtering algorithm, but each peer recursively combines partial preference vectors and stores only an aggregated version. However, an aggregate of the partial preference vectors introduces noise in the multi-cluster regime, so the performance of the multi-cluster aware global list recommendation rule is not as good as the other recommendation rules when peers in different clusters are correlated. Finally, for the accumulation of preference information, we found that if the preferences of peers in different clusters are nearly independent, either exponential accumulation of partial preference vectors or neighbor assignments such that most neighbors have similar preferences is sufficient for near optimal performance.

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